M 408L Practice Exam 3 Solutions

1. Conditionally Converges; Get in the habit of noticing that $\cos(n\pi) = (-1)^n$ (if you ever forget this fact, start plugging in $n = 1$, $n = 2$, $n = 3$, etc., to find the pattern). From here, if you check for absolute convergence, you will get

$$\left|\cos(k\pi) \cdot \frac{k^3}{(k+5)^4 \ln(k)}\right| = \frac{k^3}{(k+5)^4 \ln(k)} \geq \frac{k^3}{6k^4 \ln(k)} = \frac{1}{6k \ln(k)}$$

whenever $k \geq 2$ (here, we are using the fact that $k + 5 \leq 6k$, which is what allows us to get the right inequality). From here, use the integral test to find that

$$\sum_{k=2}^{\infty} \frac{1}{6k \ln(k)}$$

diverges. Thus, the series

$$\sum_{k=1}^{\infty} \left|\cos(k\pi) \cdot \frac{k^3}{(k+5)^4 \ln(k)}\right|$$

diverges by the comparison test, so it does *not* absolutely converge. To check for conditional convergence, use the alternating series test. You will find that

$$\lim_{k \to \infty} \frac{k^3}{(k+5)^4 \ln(k)} = 0$$

since the denominator has a higher order of k. Thus, the series conditionally converges by the alternating series test.

2. The series diverges; Use the comparison test to remove the "weaker" terms from the series:

$$\frac{3n^4}{6n^5 \ln(n) + 7} \geq \frac{3n^4}{6n^5 \ln(n) + 7n^5 \ln(n)} = \frac{3n^4}{13n^5 \ln(n)} = \frac{3}{13n \ln(n)}$$

whenever $n \geq 2$. Again, the integral test tells us $\sum_{n=2}^{\infty} \frac{3}{13n \ln(n)}$ diverges, so our original series diverges as well (note that our original sum starts at $n = 1$, but the series we are comparing to starts at $n = 2$. The comparison test still applies in this case – put simply, we do not care about whether we start at $n = 1$ or $n = 2$ since their contribution is only finite).

3. The series converges; Use the fact that $\ln(n) \leq n^{1/2}$ (our intuition here is that logarithmic functions grow *very* slowly):

$$\frac{(\ln(m))^2}{(m^3+5)} \leq \frac{(m^{1/2})^2}{(m^3+5)} = \frac{m}{m^3+5} \leq \frac{m}{m^3} = \frac{1}{m^2}.$$

But $\sum_{m=1}^{\infty} \frac{1}{m^2}$ converges (it's a p-series with $p = 2$), so our original series converges as well.

4. $C < A < B$; You can draw any positive, continuous, decreasing function for f. In this case, C is represented pictorially by right-Riemann sums and B is represented by left-Riemann sums. You will find that C underestimates the area under the curve (this is what A represents), and B overestimates the area.

5. The series diverges; Use the comparison test:

$$\frac{n^{1/3}+5}{\sqrt{n^2+50n}} \geq \frac{n^{1/3}}{\sqrt{n^2+50n^2}} = \frac{n^{1/3}}{\sqrt{51}\cdot n} = \frac{1}{\sqrt{51}} \cdot \frac{1}{n^{2/3}}.$$

The series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{51}} \cdot \frac{1}{n^{2/3}}$ diverges (it's a p-series with $p = 2/3$), so our original series diverges.

6. The series converges absolutely; Use the ratio test. If $a_n = (-1)^{n+1}\frac{n!}{n^n}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n.$$

From here, we will need to use this key fact:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

We will have to manipulate $\left(\frac{n}{n+1}\right)^n$ slightly to achieve this limit:

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^n\right]^{-1} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = e^{-1}.$$

However, $e^{-1} < 1$, so by the ratio test, our series converges absolutely.

7. The series diverges; We need nothing more than the test for divergence. Let's look at

$$\lim_{n\to\infty} \tan^{-1}\left(\frac{n}{n+2}\right).$$

This has the appearance of a "sequence plugged into a function". To find the limit, we need to find the limit of the sequence inside, then plug that limit into the function $\tan^{-1}$. We find that

$$\lim_{n\to\infty}\frac{n}{n+2} = 1,$$

and so

$$\lim_{n\to\infty} \tan^{-1}\left(\frac{n}{n+2}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$

So, the alternating sequence

$$(-1)^n \tan^{-1}\left(\frac{n}{n+2}\right)$$

cannot converge to 0. This means our series diverges.

8. The series converges absolutely; Notice that the series alternates, the numerators increase by 2, and the denominators increase by 3. With these facts, you should get that the given series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2n}{3n+1}\right)^n.$$

You will find that the limit of $(|a_n|)^{1/n}$ is $\frac{2}{3}$, so the series converges by the root test.

9. $\sum_{n=0}^{\infty}(-1)^n \cdot \frac{2^{4n+3}}{(2n+1)!(4n+3)}$; Use the power series of $\sin(x)$ as inspiration:

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

So

$$\sin(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

From here, you can integrate $\sin(x^2)$ by integrating each term of the power series:

$$\begin{aligned}\int_0^2 \sin(x^2)\,dx &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^2 x^{4n+2}\,dx \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left[\left.\frac{x^{4n+3}}{4n+3}\right|_{x=0}^{2}\right] \\ &= \sum_{n=0}^{\infty}(-1)^n \cdot \frac{2^{4n+3}}{(2n+1)!(4n+3)}\end{aligned}$$

10. $\sum_{n=0}^{\infty}(-1)^n\left[\frac{2x^{2n}}{(2n)!}+\frac{4(3x)^{2n+1}}{(2n+1)!}\right]$; Write down the power series of $2\cos(x)$ and $4\sin(3x)$:

$$2\cos(x) = \sum_{n=0}^{\infty}2(-1)^n \frac{x^{2n}}{(2n)!},$$

and

$$4\sin(3x) = \sum_{n=0}^{\infty} 4(-1)^n \frac{(3x)^{2n+1}}{(2n+1)!}.$$

From here, we can combine the two into a single power series:

$$2\cos(x) + 4\sin(3x) = \sum_{n=0}^{\infty} (-1)^n \left[\frac{2x^{2n}}{(2n)!} + \frac{4(3x)^{2n+1}}{(2n+1)!}\right] \tag{1}$$

11. Interval of convergence is $(-6, -4)$ and the radius of convergence is 1; After using the ratio test, you will find that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{15} |x+5| = |x+5|.$$

We need this to be less than 1 in order to converge, so x must be between -6 and -4. From here, check the endpoints. At $x = -4$, the series

$$\sum_{n=1}^{\infty} n^{15} \cdot (1)^n$$

diverges by the test for divergence. The same applies to when we try $x = -6$:

$$\sum_{n=1}^{\infty} n^{15} \cdot (-1)^n$$

also diverges. This means -4 and -6 are *not* in the interval of convergence, so our interval of convergence is exactly $(-6, -4)$. The radius of convergence is found by taking the length of this interval and cutting it in half.

12. Interval of convergence is $\left(-\frac{3}{2}, \frac{5}{2}\right)$ and the radius of convergence is 2; Use the ratio test like before:

$$\left|\frac{a_{n+1}}{n}\right| = \frac{n+1}{n} \cdot \left|\frac{2x-1}{4}\right|,$$

which converges to $\left|\frac{2x-1}{4}\right|$. In order for this to be less than 1, we need $-\frac{3}{2} < x < \frac{5}{2}$. From here, check the endpoints like on the last problem – you will find that the series diverges at $x = -\frac{3}{2}$ and $x = \frac{5}{2}$.

13. $\sum_{n=1}^{\infty} 2 \cdot (-1)^{n+1} 4^n x^{2n-1}$; Using the power series of $\ln(1+x)$ as inspiration, we get

$$\ln(1+4x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n x^{2n}}{n}.$$

To take the derivative, we can differentiate each term in this power series:

$$\frac{d}{dx}\ln(1+4x^2) = \sum_{n=1}^{\infty} \frac{d}{dx}\left((-1)^{n+1}\frac{4^n x^{2n}}{n}\right) = \sum_{n=1}^{\infty} 2\cdot(-1)^{n+1}4^n x^{2n-1}$$

14. The series is $\sum_{n=0}^{\infty} 3^n x^{3n}$ and the interval of convergence is $\left(\left(-\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3}\right)$; Use the power series of $\frac{1}{1-x}$ as inspiration:

$$\frac{1}{1-3x^3} = \sum_{n=0}^{\infty}(3x^3)^n = \sum_{n=0}^{\infty} 3^n x^{3n}.$$

To get the interval of convergence, use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = 3\,|x|^3.$$

In order for $3\,|x|^3$ to be less than 1, we need $\left(-\frac{1}{3}\right)^{1/3} < x < \left(\frac{1}{3}\right)^{1/3}$. You can check that the series does not converge at the endpoints, so our interval of convergence is $\left(\left(-\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3}\right)$

15. $-1+2\pi(x-\sqrt{\pi})^2$; If you compute the few derivatives of $f(x)$ and plug in $\sqrt{\pi}$, you get

$$f(\sqrt{\pi}) = -1,$$
$$f'(\sqrt{\pi}) = 0,$$
$$f''(\sqrt{\pi}) = 4\pi.$$

Once we have 2 nonzero terms, we can start constructing our Taylor polynomial:

$$f(\sqrt{\pi}) + f'(\sqrt{\pi})(x-\sqrt{\pi}) + \frac{f''(\sqrt{\pi})}{2!}(x-\sqrt{\pi})^2 = -1+2\pi(x-\sqrt{\pi})^2.$$

16. Conditionally converges; We can use the comparison test, but a shortcut is to recognize that the leading term in the numerator is $k^{3/2}$ and the leading term in the denominator is k^2. So, in the limit, the terms are "essentially" $\frac{k^{3/2}}{k^2} = \frac{1}{k^{1/2}}$ (note: you can use the comparison test to make this idea more formal). Nonetheless, the fact that the terms are approximately $\frac{1}{k^{1/2}}$ means that the series does not converge absolutely since $p = 1/2$. However, the terms still converge to zero, so the series converges conditionally by the alternating series test.

17. $6e$; First find the Taylor polynomial that we need. For this, you will need to compute the first couple derivatives of $f(x)$ and plug in $a = 1$:

$$f(1) = e,$$
$$f'(1) = 2e,$$
$$f''(1) = 6e.$$

So, our degree 2 Taylor polynomial centered at 1 is

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = e + 2e(x-1) + 3e(x-1)^2.$$

From here, e^4 is just $f(2)$, so plug in 2 into the above polynomial to get $e+2e+3e = 6e$.

18. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+5}}{2^{2n+1}(2n+1)}$; Use the Maclaurin series for $\tan^{-1}(x)$, then make some adjustments. In particular, you will need to replace x with $\frac{1}{2}x^3$, and then you will need to multiply by x^2 at the end.

19. The interval of convergence is $\{3\}$ (i.e., 3 is the only number we can plug in for x that will make the series converge) and the radius is 0; Using the ratio test gives us

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{50} \cdot |x-3|.$$

If $x = 3$, then the quantity above is zero, meaning the series will converge. If x is not 3, then the sequence above will diverge.