M 408L Practice Exam 2 Solutions

1. The integral diverges; taking the antiderivative of $\frac{1}{x^5}$ gives us $-\frac{1}{4x^4}$, and $\lim_{t\to 0} -\frac{1}{4t^4} = -\infty$.

2. The sequence converges to 0; split the fraction into $\frac{(-1)^{n+1}\cos(n)}{2\pi n^2} + \frac{1}{2\pi n}$. The numerator on the first term is bounded between -1 and 1, so both fractions will converge to 0 when you make n very large.

3. $\frac{25}{36}\ln|x-5| + \frac{11}{36}\ln|x+1| + \frac{1}{6(x+1)} + C$; using partial fraction decomposition should give you the form $\frac{A}{x-5} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$. Set this equal to the original integrand, then multiply both sides by $(x-5)(x+1)^2$
$$x^2 = A(x+1)^2 + B(x-5)(x+1) + C(x-5).$$
To find C, it is easiest to just plug in -1 for x. To find A and B, expand out the right hand side, subtract over the x^2, and factor to get
$$0 = (A+B-1)x^2 + (2A-4B+C)x + (A-5B-5C).$$
This equation is true for *every* value of x, so it must follow that all the coefficients are 0. In other words:
$$\begin{aligned} A+B-1 &= 0, \\ 2A-4B+C &= 0, \\ A-5B-5C &= 0. \end{aligned}$$
You will only need to use the first two equations to find what A and B are since we already know the value of C. Once you have the constants, the integral is straightforward – use ln for the first two terms, and u-sub with $u = x+1$ for the last term.

4. $(e^3-1)(e^2+1) = e^5 + e^3 - e^2 - 1$; from the region given, your integral should look like
$$\int_0^2 \int_0^3 ye^{x+y}\,dxdy.$$
The integral with respect to x is straightforward, but for the integral with respect to y, you will need to use integration by parts.

5. $\frac{1}{2}\ln(2)$; whenever we are presented with an integral with a lot of e^x's present, *it is never a bad idea to take* $u = e^x$. So, take $u = e^x$, so that $du = e^x dx = u dx$. This turns our original integral into
$$\int_2^5 \frac{1}{u^2-1}\,du$$
since $e^{2x} = (e^x)^2 = u^2$. From here, factor the denominator and use partial fractions.

6. The integral converges; get in the habit of recognizing that sin and cos are always bounded between -1 and 1. In this case, the numerator is always smaller than 1, so we can compare with $\int_3^\infty \frac{1}{x^5}\,dx$. This improper integral can now be easily computed.

7. $\frac{1}{2} - \frac{13}{3}\ln(2)$; The numerator has a power that is greater than or equal to the denominator, so we should start by doing polynomial long division. This will make the integrand equal to $x + 1 + \frac{x+7}{x^2+x-2}$. The first two terms are easy to integrate, and for the third term, notice that the denominator factors into $(x+2)(x-1)$; from here, use partial fractions.

8. 0; the curve $x = y^2$ is a parabola that's "turned over". Graphing out the region will give us the integral

$$\int_{-2}^{2} \int_{y^2}^{4} 3(x^2 + y^2)\,dxdy.$$

When you work this integral out, you (miraculously) get 0.

9. $f_{xy}(x, y) = f_{yx}(x, y) = \frac{2x}{y} + 2\ln(y) + 2$; the product rule will need to be used at some point since you have a function of y (namely, $(x + y)$) multiplied by $\ln(y)$. Note that f_{xy} and f_{yx} are the same by Clairaut's theorem.

10. The integral converges, and its value is 4; since we are given an absolute value, we can split the integral into two easier integrals:

$$\int_{-1}^{1} \frac{1}{\sqrt{|x|}} = \int_{-1}^{0} \frac{1}{\sqrt{-x}}\,dx + \int_{0}^{1} \frac{1}{\sqrt{x}}\,dx.$$

The antiderivative of $\frac{1}{\sqrt{x}}$ is $2\sqrt{x}$. Hence, the limit at 0 will exist, so our integral converges. In this case, we can directly compute that each integral evaluates to 2, so they add up to 4.

11. $\left(\frac{6}{7}\right)^{-1}$; since we are explicitly given what the domain is, we can write the integral as

$$\int_{1}^{2} \int_{0}^{\sqrt{y}} xy\,dxdy$$

(note that we switch the dx and dy since it is easier to integrate with respect to x first). From here, apply power rule to work out the integral.

12. The sequence converges to 1; First, look at the argument inside the sin function. By comparing the powers of the numerator and denominator, we see that both have 2 as their highest degree. Hence, to find the limit of the argument, we just divide the coefficients, meaning the argument inside sin will converge to $-\frac{3\pi}{2}$ as $n \to \infty$. Since sin is continuous, we can "plug in" the limit into sin to get the final answer. In this case, $\sin\left(-\frac{3\pi}{2}\right) = 1$.

13. The sequence diverges; compare with the function $f(x) = \frac{x^2}{\ln(x+1)}$. We can use L'Hôpital's rule to find out that $\lim_{x\to\infty} f(x) = \infty$, meaning $f(x)$ diverges. This means our sequence must also diverge (this fact comes from a theorem in the notes).

14. $\frac{3}{22}$; we have n appearing in an exponent, so we should manipulate the series to look like a geometric series. Using some exponent properties, the series we are given is the same as

$$\sum_{n=2}^{\infty} \left(\frac{3}{25}\right)^{n-1}.$$

This *almost* looks like a geometric series, but we need the summation to start at $n = 1$. We can achieve this using the classic "add and subtract the same number" trick:

$$\sum_{n=2}^{\infty} \left(\frac{3}{25}\right)^{n-1} = \left(\sum_{n=1}^{\infty} \left(\frac{3}{25}\right)^{n-1}\right) - 1.$$

All we did above was add the term corresponding to $n = 1$, then subtract it afterward. From here, we have $r = \frac{3}{25}$, which has absolute value less than 1, so this series converges. In particular, the sum is

$$\left(\sum_{n=1}^{\infty} \left(\frac{3}{25}\right)^{n-1}\right) - 1 = \frac{1}{1-\frac{3}{25}} - 1 = \frac{25}{22} - 1 = \boxed{\frac{3}{22}}.$$

15. $\frac{11}{24}$; draw out the region that is described. You will find that you get a triangle, where the bottom side is given by the equation $y = 1 - x$. So, we set up the integral

$$\int_0^1 \int_{1-x}^1 (x^2 + xy)\, dydx.$$

Use power rule to evaluate this integral.

16. If we can show that our original series is larger than a series that diverges, then our original series will diverge as well (intuition: if something is greater than ∞, then it must be ∞ too). In this case,

$$\frac{3n}{n^2-5} \geq \frac{3n}{n^2} = \frac{3}{n} \geq \frac{1}{n}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{3n}{n^2-5} \geq \sum_{n=1}^{\infty} \frac{1}{n}.$$

We are told that the series on the right diverges, so the series on the left must also diverge by the comparison test.

17. The integral diverges; since we are integrating over x values that are at least 1, we have $x+1 \leq 2x$. Also, $\ln(x)$ is positive if we plug in anything greater than 1, so

$$\frac{\ln(x)}{x+1} \geq \frac{\ln(x)}{2x} = \frac{1}{2} \cdot \frac{\ln(x)}{x}.$$

From here, we can directly get the antiderivative of this function – it is $\frac{1}{4}(\ln(x))^2$. But now, $\lim_{t\to\infty} \frac{1}{4}(\ln(t^2))^2 = \infty$, so our original integral diverges by the comparison test.

18. $\int_2^5 \int_0^{5-y} 6\sin(e^{xy})\,dxdy$; Start by sketching the bounds of integration to find out what our region looks like. You should get a right triangle – the bottom boundary is the line $y = 2$, the left boundary is the line $x = 0$, and the third side is the diagonal line $y = 5-x$. Since we are switching the order such that x comes first, we need to think in terms of "left and right" first. In this case, our left boundary is $x = 0$, and our right boundary is the line $y = 5 - x$. We need to solve this equation for x, so the right boundary is really $x = 5 - y$. Likewise, our bounds of integration for y are found by looking at the lowest and highest y-values in our region; in this case, they are 2 and 5 respectively.

19. At point P, we have $f_x < 0$ and $f_y < 0$. At point Q, we have $f_x = 0$ and $f_y < 0$; start at P and take a *very* tiny step directly to the right to get f_x. In this case, at point P, if we step to the right, we are heading towards contours with lower values (in particular, we are moving towards the -2 contour, which is smaller than we are currently at). We get f_y similarly; start at P, and move a *very* small amount upwards. We'll notice that we are moving towards contours with smaller values. For point Q, we must notice that the contour plot looks "horizontally flat" at point Q. In this case, if we take a tiny step to the right, we actually stay on our original contour, meaning the value of f doesn't change when we make a tiny step to the right. This is the same as saying $f_x = 0$. However, f_y is negative, since moving up a little bit will move us towards a contour with a lower value.