M 408L Practice Exam 1 Solutions

The actual exam will be *much* shorter than this practice exam. This practice exam is mainly here to give you plenty of opportunity to reinforce your calculus skills.

1. **Simple(r) U-subs**. Compute the following integrals:
   (a) $-\frac{1}{3\pi}$; Use the substitution $u = 3\pi x$ and recall that $\sin(\frac{3}{2}\pi) = -1$.
   (b) $\frac{1}{12\pi}e^{12\pi x} + C$; Use the substitution $u = 12\pi x$.
   (c) $\frac{1}{4}$; Use the substitution $u = x - 5$.
   (d) $\arctan(x - 10) + C$; Use the substitution $u = x - 10$ and recall that $\frac{1}{1+u^2}\,du = \arctan(u) + C$.

2. Answer: π; You can work out the integral directly even if we don't know the value of c – just treat it like a constant. In this case, the integral will work out to $1 - \cos(c)$. Set this equal to 2 and solve for c to get $c = \pi$ (in fact, the more general solution is $c = (2n+1)\pi$, where n is any whole number).

3. **Fundamental Theorem of Calculus**. Find the derivative of the following functions:
   (a) $F'(x) = \sqrt{1 + \cos(x)}$; Just replace t with x.
   (b) $F'(x) = 3x^2 e^{x^6}$; Replace t with x^3 and multiply by the derivative of x^3.
   (c) $F'(x) = -\sec^2(x)\ln(|\tan(x)|)$; Remember to flip the integral first so $\tan(x)$ appears as the top bound of integration. This is why there's a minus sign in front.
   (d) $F'(x) = 0$; Here, $F(x)$ does not depend on x, so it's just a constant. No matter how scary it looks, the derivative of a constant is zero.

4. Answer: $9 + \sin(2)$; Start by integrating $v(t)$ from 0 to 2, which gives you $4 + \sin(2)$. This is the *change* in position between times 0 and 2. To get the actual position, add this to the initial position, which is 5.

5. **Advanced U-subs**. Compute the following integrals:
   (a) $\frac{1}{20}\sin(5x^4) + C$; Use the substitution $u = 5x^4$.
   (b) $3e^{\tan(x)} + C$; Use the substitution $u = \tan(x)$.
   (c) $\frac{2}{9}(5 + x^3)^{3/2} + C$; Use the substitution $u = 5 + x^3$.
   (d) $\ln(|\sin(x)|) + C$; start by writing $\cot(x) = \frac{\cos(x)}{\sin(x)}$ and use the substitution $u = \sin(x)$.
   (e) $-\frac{\cos^{2026}(x)}{2026} + C$; Use the substitution $u = \cos(x)$.

6. **Area Between Curves.** Find the area of the described regions:
   (a) $\frac{4\sqrt{2}-2}{3} - 2\ln(2) + 1$; Set up the integral $\int_1^2 (\sqrt{x} - \ln(x))\,dx$. By following the hint, you will find that $\sqrt{x}$ is the top function.
   (b) $8\ln(2) - 4$; Set up the integrals $\int_{-2\ln 2}^{0} (2 - e^{-\frac{1}{2}x})\,dx$ and $\int_0^{2\ln 2} (2 - e^{\frac{1}{2}x})\,dx$ and add up the results. We must split the integrals like this since the bottom changes halfway throughout the region. The bounds $2\ln 2$ and $-2\ln 2$ are obtained by solving $2 = e^{\frac{1}{2}x}$ and $2 = e^{-\frac{1}{2}x}$ for x.
   (c) 9; This is better thought of as a "right minus left" area problem. By sketching, our right curve is $y = x$, and our left is $y^2 = 4x + 5$. Start by solving these equations for x, then determine the y-values where these two curves meet (this is obtained by solving $y^2 = 4y + 5$ for y). From here, you should have the integral $\int_{-1}^{5} \left(y - \frac{y^2-5}{4}\right)\,dy$.
   (d) $\cos(4) - \sin(4) + 2\sqrt{2} + 1$; Following the hint, we need to find the area between $\cos(x)$ and $-\sin(x)$. On the interval $[0,4]$, $\cos(x)$ and $-\sin(x)$ will be equal at $x = \frac{3\pi}{4}$ and nowhere else. As a result, our top and bottom functions will switch at this point. This lets us set up the integrals $\int_0^{3\pi/4} (\cos(x) + \sin(x))\,dx + \int_{3\pi/4}^{4} (-\sin(x) - \cos(x))\,dx$

7. **Volumes**. Find the volume of the described solids:
   (a) $\frac{\pi}{2}(e^2 - 3)$; You will need to use washer method since our region will have a "gap" inside. The outer radius is given by e^x and the inner radius is given by 1, so our integral will look like $\pi \int_0^1 (e^x)^2 - 1^2\,dx$.
   (b) $\frac{28\pi}{5}$; Another washer problem, but you will need to use dy since the axis of rotation is vertical. In this case, our outer radius is $2 - \frac{y^2}{9}$ and our inner radius is $2 - 1 = 1$, giving us the integral $\pi \int_0^3 \left(2 - \frac{y^2}{9}\right)^2 - 1\,dy$
   (c) π; Since the axis of rotation is vertical, we must use dy. Our solid has no gap, so we can use disk method. In this case, we find our radius by solving $y = e^{x^2}$ for x, which is $x = \sqrt{\ln y}$. From here, set up the integral $\pi \int_1^e (\sqrt{\ln y})^2\,dy$.

8. **Integration by Parts**. Compute the following integrals:
   (a) $\frac{1}{\pi} - \frac{4}{\pi^3}$; Use integration by parts twice. For the first round, $u = x^2$ and $dv = \sin(\pi x)$. For the second round, $u = x$ and $dv = \cos(\pi x)$.

(b) $-\dfrac{2e^{3x}\cos(2x)}{13}+\dfrac{3e^{3x}\sin(2x)}{13}+C$; You will need multiple rounds of integration by parts. Each time, use $dv=e^{3x}$, and u will be a trigonometric function. After a couple applications, you will cycle back to the original integral; move it to the other side and solve for the original integral.

(c) $x\ln^2(x)-2(x\ln(x)-x)+C$; You don't have too many options for what u and dv are. In this case, setting $u=$ and $dv=$ will work.

(d) $-\dfrac{5}{16e^4}+\dfrac{1}{16}$; Use $u=\ln(x)$ and $dv=x^{-5}$.

9. **Advanced Trig Integrals**. Compute the following integrals:

(a) $\dfrac{1}{8}\left(x-\dfrac{1}{4}\sin(4x)\right)+C$; The trig identity $\sin(2x)=2\sin(x)\cos(x)$ and the Pythagorean identity $\sin^2(x)+\cos^2(x)=1$ may be useful.

(b) $\dfrac{1}{3}\left(-\dfrac{\cos^3(3x)}{3}+\dfrac{\cos^5(3x)}{5}\right)+C$; Pull out a $\sin^2(3x)$ and use the Pythagorean identity. From here, u-sub with $u=\cos(3x)$.

(c) $-\dfrac{1}{2}\cos^4(x)+C$; Start by using the double angle formula $\sin(2x)=2\sin(x)\cos(x)$, then u-sub with $u=\cos(x)$.

(d) $\dfrac{\tan^6(x)}{6}+\dfrac{\tan^8(x)}{8}+C$; you can either factor out $\sec^2(x)$ and apply Pythagorean identity to the remaining $\sec^2(x)$, or pull out $\sec(x)\tan(x)$ and apply Pythagorean identity to the remaining $\tan^4(x)$.

(e) $\dfrac{1}{2}\left(-\dfrac{1}{7}\cos(7x)+\cos(x)\right)+C$; Use the identity $\cos(A)\sin(B)=\dfrac{\sin(A+B)+\sin(A-B)}{2}$ with $A=4x$ and $B=3x$.

10. **Trig Substitution**. Compute the following integrals:

(a) $\dfrac{1}{2}\left(\arcsin(x)-\dfrac{1}{2}\sin(2\arcsin(x))\right)+C$; Use the substitution $x=\sin(\theta)$.

(b) $-2\,\mathrm{arcsec}\left(\dfrac{5}{2}x\right)+\sqrt{25x^2-4}+C$; Use the substitution $x=\dfrac{2}{5}\sec(\theta)$ to force the square root to contain $\sqrt{\sec^2(\theta)-1}$.

(c) $-\dfrac{\sqrt{1+x^2}}{x}+C$; Use the substitution $x=\tan(\theta)$.

(d) $\dfrac{-9\sqrt{-x^2+9}-2x^2\sqrt{9-x^2}}{243x^3}+C$; Use the substitution $x=3\sin(\theta)$.

11. **Freestyle**. Compute the following integrals by any means necessary. Try to use the easiest method(s) available!

(a) 0; the function we're integrating is odd (cos is even, sin is odd, and x^{100} is even).

(b) 75; take advantage of the fact that $|x|$ is even to rewrite the problem as $2\int_0^5 3x\,dx$.

(c) $\frac{x^2}{2} - \frac{1}{2}\ln\left|x^2+1\right| + C$; technically, you can use trig sub with $x = \tan(\theta)$, but polynomial long division makes this much easier. Using long division, we will find that $\frac{x^3}{1+x^2} = x - \frac{x}{x^2+1}$. The second term can be integrated with u-sub.

(d) $x - \arctan(x) + C$; Again, trig-sub is possible here, but the easier approach is to notice that $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$. The antiderivative of the second term is simply arctan.

(e) $\frac{1}{2}\left(e^{x^2}x^2 - e^{x^2}\right) + C$; This problem requires a non-obvious integration by parts. Following the hint, we should set $u = x^2$ and $dv = xe^{x^2}$. You will only need to apply IBP once since $\int vdu$ can be solved with a single substitution.

(f) $\frac{5}{3}\arctan\left(x^3\right) + C$; A non-obvious u-sub. Taking $u = x^3$ is the best choice since it allows us to cancel the x^2 in the numerator, leaving us with $\frac{5}{3}\int \frac{1}{1+u^2}\,du$.