UT Austin Applied I Prelim Solutions

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This document contains solutions to the Applied I prelim problems from August 2015 to August 2025. Here, you will also find common strategies and important things to know for the prelim, along with proofs from the textbook that may show up on future prelims. The book we use is *Functional Analysis for the Applied Mathematician* by Todd Arbogast and Jerry Bona.

Acronyms

- PUB: Principle of Uniform Boundedness (also called the Uniform Boundedness Principle)
- HBT: Hahn-Banach Theorem (this acronym will *never* be used to refer to the Heine-Borel Theorem)
- OMT: Open Mapping Theorem
- CGT: Closed Graph Theorem
- BAT: Banach-Alaoglu Theorem
- RRT: Riesz Representation Theorem
- ST(c): Spectral Theorem for Compact Operators:
- ST(sa): Spectral Theorem for Self-Adjoint Operators
- ST(csa): Spectral Theorem for Compact, Self-Adjoint Operators
- MCT: Monotone Convergence Theorem (the one for integrals, not sequences of real numbers)
- DCT: Dominated Convergence Theorem

General Advice

"Primers" in Multi-Part Problems

If a part (a) tells you to state or prove a well-known result from the book, it is nearly guaranteed that you are expected to use it on part (b) (for example, the writers of these exams love asking you to state PUB or CGT, then having you use it on the next part).

Showing Continuity

If you are being asked to show continuity of an operator between *Banach* spaces, consider using CGT (do not use this if the domain or codomain is not complete). Typically, when showing something is continuous, an analyst follows these two steps:

1. Assume $x_n \to x$.
2. Show that $Tx_n \to Tx$.

When using the closed graph theorem, we want to verify that Graph(T) is closed. This means assuming (x_n, Tx_n) is a sequence in Graph(T) converging to (x, y), then showing that $(x, y) \in$ Graph(T), i.e., $y = Tx$. In other words, the pipeline for showing continuity via the closed graph theorem is:

1. Assume $x_n \to x$.
2. *Additionally assume* $Tx_n \to y$.
3. Show that $Tx = y$.

In many cases, this strategy is easier than the first one.

Of course, since we are interested in linear operators in Applied I, another way of showing continuity is by showing boundedness. This can be shown in multiple (equivalent) ways:

1. Show the existence of a constant $C \geq 0$ such that for all x with $\|x\| \leq 1$, $\|Tx\| \leq C$.
2. Show the existence of of a constant $C \geq 0$ such that for all x, $\|Tx\| \leq C\|x\|$.
3. Show the existence of a constant $C \geq 0$ such that for all $x \neq 0$, $\dfrac{\|Tx\|}{\|x\|} \leq C$.

Showing boundedness does not require our space to be Banach. Often, we do not know beforehand what this constant C is, so you might find it useful to just begin naively estimating $\|Tx\|$ using whatever information the problem gives you.

Showing Two Things are Equal

In classical analysis, a common way to show two real numbers x and y are equal is by showing $x \leq y$ and then $y \leq x$. This may also be useful in Applied I. However, the language of functional analysis gives us several more practical ways to show things are equal; rather than directly comparing two objects, we compare how these objects interact with functionals (i.e., comparing these objects in a "weak" sense).

In particular, in a normed linear space X, $x = y$ if and only if $f(x) = f(y)$ for all $f \in X^*$ (recall that this follows from a corollary of Hahn-Banach: linear functionals separate points). Similarly, in a Hilbert space H, $x = y$ if and only if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in H$. To recap, when showing $x = y$, the most common methods are

- Classical: Show $x \leq y$ then $y \leq x$. Setting: real numbers
- Complex Classical: Show $\Re(x) = \Re(y)$ and $\Im(x) = \Im(y)$. Setting: complex numbers
- Strong Sense: Show $\|x - y\| = 0$ (or, alternatively, show $\|x - y\| < \varepsilon$ for all $\varepsilon > 0$). Setting: normed linear spaces
- Weak Sense: Show $f(x) = f(y)$ for all $f \in X^*$. Setting: normed linear spaces
- Inner Product Sense: Show $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in H$. Setting: inner product spaces

For the inner product sense, you actually only need to show $\langle x, u_\alpha\rangle = \langle y, u_\alpha\rangle$ for all u_α in some ON basis (this follows from Parseval's theorem).

Separating Points from Subspaces

Key intuition: linear functionals can be utilized geometrically. If you have a closed subspace M and a point $x \notin M$, there is a feeling that there should be some "gap" between x and M. Linear functionals allow us to make this idea concrete; in particular, we can use Mazur's first separation lemma. In this case, the lemma says there exists $f \in X^*$ such that $f(x) > 0$ and $f|_M \equiv 0$. Having such a linear functional in play is often quite useful (several problems below will use this lemma to get clean proofs). Other results that use linear functionals geometrically are Mazur's second separation lemma and the separating hyperplane theorem (FYI, all of these things follow from Hahn-Banach).

Obtaining a Limit

Often, desirable objects in functional analysis are not constructed explicitly, but are rather obtained by taking a limit of some sequence. This is why we often choose to study Banach and Hilbert spaces; in these settings, once we know a sequence is Cauchy, we know it has a limit in our space. The prime example of when this is used is in showing the existence of a best approximation in a Hilbert space (recall that the strategy is to look at a sequence that gets "arbitrarily close" to the minimum distance, showing that sequence is Cauchy, then taking its limit).

We also have tools to obtain limits without having to prove Cauchy-ness. For example, in finite-dimensional spaces, the Bolzano-Weierstrass theorem tells us that bounded sequences have convergent subsequences. Often, we do not work in finite-dimensional spaces, but we can upgrade to the infinite-dimensional setting at the cost of replacing strong convergence with weak convergence by using the Banach-Alaoglu theorem. Whenever you are working in a reflexive, separable Banach space (i.e., the most relevant Banach spaces), you can obtain weak limits simply by constructing a bounded sequence; often, this weak limit will be precisely what you need to complete a problem. Several of the problems below will use this strategy; it is particularly useful when showing existence of a solution to a problem like $Tx = f$, where T is a linear operator.

August 2025

Problem 1

(a)

Suppose $y, z \in H$ satisfy $Lx = \langle x, y\rangle = \langle y, z\rangle$ for all $x \in H$. Testing with $x = y - z$, we get $\|y - z\|^2 = \langle y - z, y - z\rangle = 0$, so $y = z$.

(b)

If $\ker(L) = H$, then clearly we take $y = 0$. Otherwise, $(\ker(L))^\perp$ is nonzero, so we can take a unit vector $z \in (\ker(L))^\perp$.

<u>Trick</u>: Define $y := (\overline{Lz})z$. We find

$$\langle x, y\rangle = \langle (Lz)x, z\rangle = \langle (Lz)x - (Lx)z, z\rangle + \langle (Lx)z, z\rangle = \langle (Lx)z, z\rangle = Lx.$$

since $L((Lz)x - (Lx)z) = 0$, and so $(Lz)x - (Lx)z \perp z$.

(c)

If $L \equiv 0$, then $y = 0$, making the result hold trivially. So, suppose $y \neq 0$. First, apply Cauchy-Schwarz to get

$$|Lx| = |\langle x, y\rangle| \leq \|x\| \, \|y\| \,, \quad \forall x \in H,$$

establishing $\|L\| \leq \|y\|$. For the other inequality:

$$\|L\| \geq \left| L\left(\frac{y}{\|y\|}\right)\right| = \frac{1}{\|y\|} \left|\langle y, y\rangle\right| = \|y\| \,.$$

Problem 2

(a)

X is separable if it contains a countable, dense, subset.

(b)

X^* is the normed linear space of continuous linear functionals from X to the field $\mathbb{F}$ equipped with the operator norm.

(c)

Let $\{f_n\}_{n=1}^{\infty}$ be a countable, dense subset of X^*. For all n, we can find $x_n \in X$ such that $\|x_n\| = 1$ and $|f_n(x_n)| \geq \dfrac{1}{2}\|f_n\|$. Let

$$\mathcal{D} := \left\{ \sum_{i=1}^{n} q_i x_i : q_i \in \mathbb{Q} \right\},$$

which is countable. We claim that $\mathcal{D}$ is dense in X. Clearly, $\overline{\mathcal{D}}$ is a subspace (in fact, it's just the span of $\{x_n\}_{n=1}^{\infty}$). Suppose for the sake of contradiction that there exists $x \notin \overline{\mathcal{D}}$.

<u>Trick</u>: By Mazur's first separation lemma, there exists $f \in X^*$ such that $f(x) > 0$ but $f|_{\overline{\mathcal{D}}} \equiv 0$ (it's important that $\overline{\mathcal{D}}$ is a closed, linear subspace of X to use this lemma).

In particular, $f(x_n) = 0$ for all $n \in \mathbb{N}$. Now, $\{f_n\}_{n=1}^{\infty}$ is dense in X^*, so there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ converging to f. We conclude

$$\begin{aligned} \|f_{n_k}\| &\leq 2\left|f_{n_k}(x_{n_k})\right| = 2\left|f_{n_k}(x_{n_k}) - f(x_{n_k})\right| \leq 2\left\|f_{n_k} - f\right\|\left\|x_{n_k}\right\| \\ &= 2\left\|f_{n_k} - f\right\| \to 0. \end{aligned}$$

Hence, $f_{n_k} \to 0$, so f must be zero. This leads to a contradiction since $f(x) > 0$, so $\overline{\mathcal{D}} = X$.

Problem 3

Let $M := \sup_n \lambda_n < \infty$.

(a)

We must first confirm that the series defining A converges for all $x \in H$. Let $x \in H$ and let S_n denote the n-th partial sum of Ax. If $N > M$, then

$$\|S_N - S_M\|^2 = \left\| \sum_{n=M+1}^{N} \lambda_n \langle x, e_n \rangle e_n \right\|^2 = \sum_{n=M+1}^{N} \left|\lambda_n \langle x, e_n \rangle\right|^2 \leq M^2 \sum_{n=M+1}^{N} \left|\langle x, e_n \rangle\right|^2.$$

by orthonormality. Now, $\sum_{n=1}^{\infty} \left|\langle x, e_n \rangle\right|^2$ converges by Parseval's identity, so the right hand side converges to 0 as $N, M \to \infty$. Hence, $\{S_n\}_{n=1}^{\infty}$ is Cauchy, so the series defining Ax does converge. Linearity is easy now that we know these series converge:

$$A(\alpha x + y) = \sum_{n=1}^{\infty} \lambda_n \langle \alpha x + y, e_n \rangle e_n = \alpha \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n + \sum_{n=1}^{\infty} \lambda_n \langle y, e_n \rangle e_n.$$

For boundedness, we find

$$\|Ax\|^2 = \sum_{n=1}^{\infty} |\lambda_n \langle x, e_n \rangle|^2 \leq M^2 \|x\|^2$$

by applying Parseval's again, so $\|A\| \leq M$. Last, we check self-adjointness:

$$\begin{aligned} \langle Ax, y \rangle &= \sum_{n=1}^{\infty} \langle \lambda_n \langle x, e_n \rangle e_n, y \rangle \\ &= \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \overline{\langle y, e_n \rangle} \\ &= \sum_{n=1}^{\infty} \langle x, \lambda_n \langle y, e_n \rangle e_n \rangle && \lambda_n \text{ is real} \\ &= \langle x, Ay \rangle, \end{aligned}$$

where the continuity of $\langle \cdot, \cdot \rangle$ lets us pass inner products into the sum.

(b)

Clearly, $\sigma_P(A) = \{\lambda_n : n \in \mathbb{N}\}$ (remember: these eigenvalues need not be distinct!). In particular, the eigenspace of each distinct eigenvalue λ_n is the span of $\{e_m : m \in \mathbb{N}, \lambda_m = \lambda_n\}$.

(c)

If all the λ_n are nonzero and are bounded away from zero (i.e., $\inf_n \lambda_n > 0$), then A is surjective. Remember that $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ for all $x \in H$ by Riesz-Fischer. So, for all $y \in H$,

$$y = \sum_{n=1}^{\infty} \lambda_n \left\langle \frac{y}{\lambda_n}, e_n \right\rangle e_n.$$

This leads to us defining $x := \sum_{n=1}^{\infty} \left\langle \frac{y}{\lambda_n}, e_n \right\rangle e_n$. By our assumption, $\frac{1}{\lambda_n}$ stays bounded in n. Hence, this element x is well-defined (apply the same idea from part (a)), and clearly $Ax = y$.

(c)

Define the partial sums $A_n x := \sum_{k=1}^{n} \lambda_k \langle x, e_k \rangle e_k$. Each A_n is a compact operator since they are bounded with finite-dimensional range. If we can show that $A_n \to A$ in the operator norm, we are done. Indeed, for any $\varepsilon > 0$, if we choose $N \in \mathbb{N}$ such that $\lambda_n < \varepsilon^2$ for all $n \geq N$, then for all $x \in H$,

$$\|A_n x - Ax\|^2 = \left\| \sum_{k=n+1}^{\infty} \lambda_k \langle x, e_k \rangle e_k \right\|^2 = \sum_{k=n+1}^{\infty} |\lambda_k \langle x, e_k \rangle|^2 \leq \varepsilon^2 \sum_{k=n+1}^{\infty} |\langle x, e_k \rangle|^2 \leq \varepsilon^2 \|x\|^2$$

by Bessel's inequality. Thus, $\|A_n - A\| < \varepsilon$, so $A_n \to A$.

January 2025

Problem 1

$\underline{\implies}$ Notice that for all $\alpha \in (0,1)$ and $y, z \in S$, we have $y + \alpha(z - y) = \alpha z + (1-\alpha)y \in S$. Hence,

$$\begin{aligned} \|x - y\|^2 &\leq \|x - [y + \alpha(z - y)]\|^2 \\ &= \|x - y\|^2 - 2\alpha \langle x - y, z - y \rangle + \alpha^2 \|z - y\|^2 \qquad \text{the field is real.} \end{aligned}$$

Cancel out an α to get

$$\langle x - y, z - y \rangle \leq \frac{\alpha}{2} \|z - y\|^2,$$

and sending $\alpha \to 0$ gives $\langle x - y, z - y \rangle \leq 0$.

$\underline{\impliedby}$ We directly compute

$$\begin{aligned} \|x - z\|^2 &= \|(x - y) + (y - z)\|^2 \\ &= \|x - y\|^2 - 2 \langle x - y, z - y \rangle + \|y - z\|^2 \\ &\geq \|x - y\|^2 + \|y - z\|^2 \\ &\geq \|x - y\|^2. \end{aligned}$$

Problem 2

Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in X. Since T_1 is compact, $\{T(x_n)\}_{n=1}^{\infty}$ has a convergent subsequence that we will call $\{T(x_{1,n})\}_{n=1}^{\infty}$. From the bounded subsequence $\{x_{1,n}\}_{n=1}^{\infty}$, we can extract another subsequence $\{x_{2,n}\}_{n=1}^{\infty}$ such that $T_2(x_{2,n})$ converges. Now, continue inductively to get nested subsequences $\{x_{1,n}\}_{n=1}^{\infty} \supseteq \{x_{2,n}\}_{n=1}^{\infty} \supseteq \ldots$ such that $\{T_m(x_{m,n})\}_{n=1}^{\infty}$ converges for all $m \in \mathbb{N}$. From here, we apply our diagonalization argument: define $\tilde{x}_n = x_{n,n}$ for all n.

We claim $T(\tilde{x}_n)$ is Cauchy in Y, which is enough to complete the proof. Indeed, if $n \geq m$,

$$\begin{aligned}
\|T(\tilde{x}_n) - T(\tilde{x}_m)\| &\leq \|T(\tilde{x}_n) - T_m(\tilde{x}_n)\| + \|T_m(\tilde{x}_n) - T_m(\tilde{x}_m)\| + \|T_m(\tilde{x}_m) - T(\tilde{x}_m)\| \\
&\leq \|T - T_m\| \cdot \|\tilde{x}_n\| + \|T - T_m\| \cdot \|\tilde{x}_m\| + \|T_m(\tilde{x}_n) - T_m(\tilde{x}_m)\| .
\end{aligned}$$

The first two terms on the right hand side go to 0 as $n, m \to \infty$ since $\{T_n\}_{n=1}^{\infty}$ converges to T in the operator norm and $\{\tilde{x}_n\}_{n=1}^{\infty}$ is bounded. The third term also tends to 0 as $n, m \to \infty$; this is because $n \geq m$ means $\tilde{x}_n$ is in the subsequence $\{x_{m,n}\}_{n=1}^{\infty}$.

Problem 3

It's worth to check that $\mathrm{PV}(1/x)$ is well-defined in the sense that the limit exists for all $\phi \in \mathcal{D}$. Indeed,

$$\begin{aligned}
|\langle \mathrm{PV}(1/x), \phi \rangle| &= \left| \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{1}{x} \phi(x)\, dx \right| \\
&= \lim_{\varepsilon \to 0^+} \left| \phi(x) \ln|x| \, \Big|_{|x|>\varepsilon} - \int_{|x|>\varepsilon} \ln|x| \, \phi'(x)\, dx \right| \\
&= \lim_{\varepsilon \to 0^+} \left| -\ln(\varepsilon)[\phi(\varepsilon) - \phi(-\varepsilon)] - \int_{|x|>\varepsilon} \ln|x| \, \phi'(x)\, dx \right| .
\end{aligned}$$

Take the limit of the boundary terms:

$$\lim_{\varepsilon \to 0^+} |\ln(\varepsilon)[\phi(\varepsilon) - \phi(-\varepsilon)]| = \lim_{\varepsilon \to 0^+} \left| 2\varepsilon \ln(\varepsilon) \cdot \left[\frac{\phi(\varepsilon) - \phi(-\varepsilon)}{2\varepsilon} \right] \right| = 0$$

since the term in brackets converges to $\phi'(0)$ and $2\varepsilon \ln(\varepsilon) \to 0$. For the integral, recall that $\ln|x|$ is locally integrable and $\phi'(x)$ is bounded and compactly supported, so by the dominated convergence theorem,

$$\lim_{\varepsilon \to 0^+} \left| \int_{|x|>\varepsilon} \ln|x| \, \phi'(x)\, dx \right|$$

converges. From this, linearity is easy to check. To show boundedness, fix $K \subseteq\subseteq \mathbb{R}$ and let $\phi \in \mathcal{D}_K$. We find

$$\begin{aligned}
|\langle \mathrm{PV}(1/x), \phi \rangle| &= \lim_{\varepsilon \to 0^+} \left| \int_{|x|>\varepsilon} \ln|x| \, \phi'(x)\, dx \right| = \left| \int_K \ln|x| \, \phi'(x)\, dx \right| \\
&\leq \|\phi'\|_{\infty} \left| \int_K \ln|x| \; dx \right| \leq \|\phi\|_{1,\infty,\mathbb{R}} \left| \int_K \ln|x| \; dx \right| ,
\end{aligned}$$

where the integral on the right is finite since $\ln|x|$ is locally integrable.

Problem 4

(a)

For each n, there exists $x_n \in A_n$. We claim that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Indeed, if $n \geq m$, then $x_n, x_m \in A_m$, so

$$d(x_n, x_m) \leq \operatorname{diam}(A_m) \to 0.$$

Because X is complete, $x_n \to x$ for some $x \in X$. Each A_n is closed as well, so $x \in A_n$ for all $n \in \mathbb{N}$, i.e., $x \in \bigcap_{n=1}^{\infty} A_n$. To show that x is the *only* element in this intersection, suppose $y \in \bigcap_{n=1}^{\infty} A_n$. Then, for all $n \in \mathbb{N}$, $x, y \in A_n$, hence $d(x, y) \leq d(A_n)$. But, $d(A_n) \to 0$, so $d(x, y) = 0$.

(b)

Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then, there exists $N_1 \in \mathbb{N}$ and a closed ball B_1 of radius 1 such that $x_n \in B_1$ for all $n \geq N_1$. Likewise, there exists $N_2 \geq N_1$ and a closed ball B_2 of radius $1/2$ such that $x_n \in B_2$ for all $n \geq N_2$.

From here, take $A_n := \bigcap_{k=1}^{n} B_k$ for all n. This gives us a sequence of nested closed, nonempty sets $A_1 \supseteq A_2 \supseteq \ldots$ with diameters converging to 0. Hence, $\bigcap_{n=1}^{\infty} A_n = \{x\}$ for some $x \in X$. We claim $x_n \to x$.

Notice that for all $k \in \mathbb{N}$, if $n \geq N_k$, then $x_n \in A_k$. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that $\operatorname{diam}(A_k) < \varepsilon$. Then, if $n \geq N_k$, we have $x_n, x \in A_k$, meaning $d(x_n, x) < \varepsilon$.

August 2024

Problem 1

(a)

T is closed if $\operatorname{Graph}(T) := \{(x, Tx) : x \in D\}$ is topologically closed in $X \times Y$.

(b)

Let X, Y be Banach spaces. An operator $T : X \to Y$ is bounded if and only if it is closed.

(c)

$\underline{\implies}$ Let T be bounded and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in D converging to $x \in X$. Since T is continuous, Tx_n converges to some element $y \in Y$. In particular, $(x_n, Tx_n) \to (x, y)$. Now, since T is closed, $\operatorname{Graph}(T)$ is closed, so $(x, y) \in \operatorname{Graph}(T)$, meaning $x \in D$, i.e., D is closed.

$\underline{\Longleftarrow}$ Suppose D is closed. Then, D is a Banach space in its own right, so by the closed graph theorem, T is bounded on D.

NB: For the forward implication, we cannot say $Tx_n \to Tx$ at the start since, at this point, we do not know if $x \in D$ (i.e., we don't yet know if we are allowed to plug x into T). This is why we introduce this element $y \in Y$ instead (although, by the end of the proof, we do know that $y = Tx$).

Problem 2

(a)

Let X be a Banach space, Y a normed linear space, and $\{T_\alpha\}_{\alpha \in \mathcal{I}}$ a collection of bounded linear operators from X to Y. Then, $\{T_\alpha\}_{\alpha \in \mathcal{I}}$ is uniformly bounded in operator norm if and only if it is pointwise bounded.

(b)

For all $x \in X$, $\{T_n x\}_{n=1}^\infty$ is Cauchy in Y and therefore bounded. Hence, $\{T_n\}_{n=1}^\infty$ is uniformly bounded in operator norm by PUB.

(c)

Since Y is Banach, $\{T_n x\}_{n=1}^\infty$ has a unique limit for all $x \in X$. Hence, T is well-defined. For linearity, notice that

$$T(\alpha x + y) = \lim_{n\to\infty} T_n(\alpha x + y) = \lim_{n\to\infty} [\alpha T_n x + Ty] = \alpha Tx + Ty$$

since all of the above limits exist. For boundedness, compute

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \le \liminf_{n\to\infty} \|T_n\| \, \|x\| \le \left(\sup_n \|T_n\|\right) \|x\| ,$$

and so $\|T\| \le \sup_n \|T_n\|$, which is finite by part (b).

Problem 3

(a)

By Parseval's theorem,

$$\sum_{k=n}^\infty |\langle x, e_k\rangle|^2$$

converges for all $n \in \mathbb{N}$, and so

$$\|P_n x - x\|^2 = \left\| \sum_{k=n+1}^\infty \langle x, e_k\rangle e_k \right\|^2 = \sum_{k=n+1}^\infty |\langle x, e_k\rangle|^2 \to 0$$

as $n \to \infty$. Hence, $P_n x \to x$.

(b)

From part (a), we see that $\{f_n\}_{n=1}^{\infty}$ is bounded in n. Thus, $\{x_n\}_{n=1}^{\infty}$ is also bounded in n by assumption. Because H is a separable Hilbert space, $\{x_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ by Banach-Alaoglu. Let x be the weak limit of this subsequence. We claim $Ax = f$.

Since A is bounded, $Ax_n \rightharpoonup Ax$. From here, whenever $n_k \geq j$, we have

$$\langle Ax_{n_k}, u_j \rangle = \langle f, u_j \rangle$$

since the orthogonal projections of Ax_{n_k} and f onto $\operatorname{span}\{u_1, \dots, u_{n_k}\}$ agree. Hence, for all $j \in \mathbb{N}$,

$$\langle Ax, u_j \rangle = \lim_{k\to\infty} \langle Ax_{n_k}, u_j \rangle = \lim_{k\to\infty} \langle f, u_j \rangle = \langle f, u_j \rangle ,$$

and so $Ax = f$, since by Parseval's theorem,

$$\|Ax - f\|^2 = \sum_{j=1}^{\infty} |\langle Ax - f, u_j \rangle|^2 = 0.$$

January 2024

Problem 1

Linearity is obvious. By Hölder's inequality,

$$\begin{aligned}
\|Af\|_q^q &= \int_U \left| \int_V K(u,v) f(v)\, dv \right|^q du \\
&\leq \int_U \left| \|K(u,\cdot)\|_q \|f\|_p \right|^q du \\
&\leq \|f\|_p^q \|K\|_q^q ,
\end{aligned}$$

which implies $\|A\| \leq \|K\|_q$.

For compactness, I believe we need to assume U and V are bounded. If this is the case, we can use Arzelà-Ascoli and the fact that continuous functions are dense in L^q. In particular, we can define a sequence $\{K_n\}_{n=1}^{\infty}$ of continuous functions on $U \times V$ that converge in L^q to K. Then, the integral operators defined by

$$A_n f(u) := \int_V K_n(u,v) f(v)\, dv$$

are compact on $C(\overline{U \times V})$ with the L^∞ norm. However, since $U \times V$ has finite measure, $\|f\|_q \leq \|f\|_\infty$ for all $f \in L^q(V)$. This, combined with the density of continuous functions in $L^p(U)$ tells us that each A_n is also compact as an operator on $L^p(U)$. Finally, repeating the computation at the start of this problem reveals that $\|A_n - A\| \leq \|K_n - K\|_q \to 0$, and so A is compact being a limit of compact operators.

Problem 2

Let X be a reflexive Banach space and $Y \subseteq X$ a closed linear subspace. Let $f \in Y^{**}$. Then, extend f by defining $\tilde{f}(g) = f(g|_Y)$ for all $g \in X^*$. Clearly, $\tilde{f} \in X^{**}$. Since X is reflexive, $\tilde{f} = E_x$ for some $x \in X$. We claim $x \in Y$.

<u>Trick</u>: Suppose $x \notin Y$. By Mazur's first separation lemma, there exists $g \in X^*$ such that $g(x) > 0$ but $g|_Y \equiv 0$. However, this means

$$g(x) = E_x(g) = \tilde{f}(g) = f(g|_Y) = 0,$$

a contradiction. Thus, $x \in Y$. Finally, for all $h \in Y^*$,

$$f(h) = \tilde{f}(h) = h(x),$$

and so $f = E_x$ (here, E_x is being thought of as an element of Y^{**}).

Problem 3

First, we should notice that $B = A^*$, and since $\|A\| = \|A^*\|$, it is enough to show that B is bounded. First, consider the collection $\{B\varphi\}_{\varphi \in X^*, \|\varphi\| \leq 1}$. This collection is pointwise bounded, since if $x \in X$ and $\varphi \in X^*$ has norm at most 1, then

$$\|(B\varphi)(x)\| = \|\varphi(Ax)\| \leq \|\varphi\| \, \|Ax\| \leq \|Ax\| \, .$$

By PUB, $\{B\varphi\}_{\varphi \in X^*, \|\varphi\| \leq 1}$ is uniformly bounded, meaning $\|B\| = \sup_{\varphi \in X^*, \|\varphi\| \leq 1} \|B\varphi\| < \infty$. Since B is bounded, so is A.

Problem 4

<u>$\implies$</u> I feel like we also need to assume A has a bounded inverse in order to prove this direction. This is because Y is not complete, meaning OMT is not directly applicable here. If this is the case, then for all $x \in X$,

$$\|x\| = \left\|A^{-1}Ax\right\| \leq \left\|A^{-1}\right\| \cdot \|Ax\| \, ,$$

meaning A is bounded below. Since $(A^*)^{-1} = (A^{-1})^*$, the same argument works for A^* as well.

<u>$\impliedby$</u> Clearly, bounded below implies one-to-one, so we only need to show that A is surjective. To this end, we will first show that $R(A)$ is closed. Suppose $Ax_n \to y$ in Y. Then, since A is bounded below,

$$\|x_n - x_m\| \leq \gamma \, \|Ax_n - Ax_m\|$$

for some $\gamma > 0$. But the right hand side converges to 0 as $n, m \to \infty$ since $\{Ax_n\}_{n=1}^{\infty}$ converges, so $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Since X is complete, $\{x_n\}_{n=1}^{\infty}$ has a limit x. But now, since A is continuous, $Ax_n \to Ax$, and so $y = Ax$, i.e., $R(A)$

is closed. From here, we claim that $R(A) = Y$.

Trick: Suppose for the sake of contradiction that there exists $y \in Y \setminus R(A)$. Then, $R(A)$ is a closed linear subspace of Y, so by Mazur's first separation lemma, there exists $f \in Y^*$ such that $f(y) > 0$ but $f|_{R(A)} = 0$. In other words, $A^*(f) = f \circ A \equiv 0$. However, A' is also one-to-one, so f must be identically 0, a contradiction. Hence, $R(A) = Y$, and so A is invertible.

Problem 5

Let $K := [-1, 1]$. Since T is a distribution, there exist $C > 0$ and $n \in \mathbb{Z}_{\geq 0}$ such that

$$|T\phi| \leq C \|\phi\|_{n,\infty,\Omega}$$

for all $\phi \in \mathcal{D}_K$. Then, let $h \in \mathcal{D}$ be 1 on $(-1, 1)$ and define $h_\varepsilon(x) := h(x/\varepsilon)$. Now, let $g \in \mathcal{D}$ satisfy $g(0) = g'(0) = \ldots = g^{(n)}(0) = 0$. Notice that for any $0 < \varepsilon < 1$, $g - gh_\varepsilon$ is identically zero on an open ball around 0. Hence, $T(g) = T(gh_\varepsilon)$ by assumption. Now, we can take ε small enough such that $\text{supp}(gh_\varepsilon) \subseteq K$, which will give us

$$|Tg| = |T(gh_\varepsilon)| \leq C \|(gh_\varepsilon)\|_{n,\infty,\Omega} = C \sum_{k=0}^{n} \left\|(gh_\varepsilon)^{(k)}\right\|_\infty .$$

Now, as $\varepsilon \to 0$, the above sum converges to $\sum_{k=0}^{n} \left|g^{(k)}(0)\right| = 0$. Hence, $Tg = 0$.

August 2023

Problem 1

For linearity, we see that

$$\varphi(f(\alpha x + y)) = \alpha\varphi(f(x)) + \varphi(f(y)) = \varphi(\alpha f(x) + f(y))$$

since $\varphi \circ f$ and φ are linear. Since this holds for all $\varphi \in X^*$, we must have $f(\alpha x + y) = \alpha f(x) + f(y)$ by Hahn-Banach.

For boundedness, use PUB on the collection $\{f^*\varphi\}_{\varphi \in Y^*, \|\varphi\| \leq 1}$ like before to get that f^* is bounded. Hence, f is bounded as well.

Problem 2

$\underline{\implies}$ Suppose P is continuous. Clearly, $N(P)$ is closed. For the range, suppose $Px_n \to y$. We see $Px_n = P^2 x_n \to Py$, and so $y = Py \in R(P)$.

$\underline{\impliedby}$ Suppose $N(P)$ and $R(P)$ are closed.

Trick: Since we're working over a Banach space, we can use the closed graph theorem. Suppose $(x_n, Px_n) \to (x, y)$, so $x_n \to x$ and $Px_n \to y$. Now, $R(P)$ being closed means $y = Pz$ for some z. We claim $Px = Pz$. Notice that $P(x_n - Px_n) = 0$ for all n, and $x_n - Px_n \to x - Pz$, meaning $x - Pz \in N(P)$ since $N(P)$ is closed. Hence, $Px = P^2 z = Pz$, which proves Graph(P) is closed.

Problem 3

Let T be a compact operator, $\{x_n\}_{n=1}^\infty$ be a sequence that weakly converges to x, and let $\{Tx_{n_k}\}_{k=1}^\infty$ be a subsequence of $\{Tx_n\}_{n=1}^\infty$. Recall that weakly convergent subsequences are bounded, and so $\{Tx_{n_k}\}_{k=1}^\infty$ has a further subsequence that converges strongly. Note that this subsequence also converges weakly to Tx since T is continuous, so by the uniqueness of weak limits, this subsequence must converge to Tx strongly. Hence, the original sequence $\{Tx_n\}_{n=1}^\infty$ converges strongly to Tx by the Urysohn subsequence principle.

Problem 4

https://math.stackexchange.com/questions/2390715/sum-of-unitary-operators-converges-to-projection-operator

(in recent years, unitary operators have not been discussed, so I doubt a problem like this would show up).

Problem 5

Fix $\phi \in \mathcal{D}$ and integrate by parts:

$$\int m^2 \sin(mx)\phi(x)\,dx = \int m\cos(mx)\phi'(x)\,dx = -\int \sin(mx)\phi''(x)\,dx.$$

Because $\phi'' \in \mathcal{D}$ as well, it suffices to show $\int \sin(mx)\phi(x)\,dx \to 0$. For this, we can use the fact that step functions are dense in L^1. Notice that if $a < b$,

$$\left|\int_a^b \sin(mx)\right|\,dx = \frac{1}{m}\left|\cos(mb) - \cos(ma)\right| \le \frac{2}{m} \to 0.$$

Hence, $\int \sin(mx)\psi(x)\,dx \to 0$ for any step function ψ (i.e., ψ is a linear combination of indicator functions of closed intervals). Now, for any $\varepsilon > 0$, there exists a step function ψ such that $\|\phi - \psi\|_1 < \varepsilon$. This gives us

$$\begin{aligned}\left|\int \sin(mx)\phi(x)\,dx\right| &\le \left|\int \sin(mx)[\phi(x) - \psi(x)]\,dx\right| + \left|\int \sin(mx)\psi(x)\,dx\right| \\ &\le \|\sin(mx)\|_\infty \|\phi - \psi\|_1 + \left|\int \sin(mx)\psi(x)\,dx\right| \\ &\le \varepsilon + \left|\int \sin(mx)\psi(x)\,dx\right|.\end{aligned}$$

The integral on the right vanishes as $m \to \infty$.

January 2023

Problem 1

(a)

The dual operator T^* is defined as $T^*(g) = g \circ T$. This is in Y^* since it is clearly linear and $\|g \circ T\| \leq \|g\| \cdot \|T\| < \infty$.

(b)

The inequality in part (a) immediately tells us $\|T^*\| \leq \|T\|$.

(c)

For all $y \in X^*$, there exists $f_y \in Y^*$ such that $\|f_y\| = 1$ and $f_y(y) = \|y\|$. It follows that for all $x \in X$,

$$\|Tx\| = \|f_{Tx}(Tx)\| = \|(T^* f_{Tx})(x)\| \leq \|T^* f_{Tx}\| \cdot \|x\| \leq \|T^*\| \cdot \|x\|,$$

establishing $\|T\| \leq \|T^*\|$.

Problem 2

(a)

Inner product is linear in the first component and continuous, so this is obvious.

(b)

Let $x \in N^\perp$ and suppose $Sx = 0$. Then, $P^\perp x \in N \cap N^\perp$, meaning $P^\perp x = 0$. Thus, $x \in N \cap N^\perp$, meaning $x = 0$.

Clearly, $R(S) \subseteq R(T)$. For the reverse, notice that $Tx = T(Px + P^\perp x) = Sx$ since $Px \in N$, meaning $R(T) \subseteq R(S)$.

Problem 3

(a)

$\phi_j \to \phi$ if there exists $K \subseteq\subseteq \Omega$ such that $\operatorname{supp} \phi_j \subseteq K$ for all j and $\|\phi_j - \phi\|_{n,\infty,\Omega} \to 0$ for all $n \in \mathbb{N}$.

(b)

$\underline{\implies}$ Proceed by contrapositive. Suppose the boundedness condition does not hold. Then, there exists some $K \subseteq\subseteq \Omega$ such that for all $n \in \mathbb{N}$, there exists $\phi_n \in \mathcal{D}_K$ such that

$$|T(\phi_n)| > n \|\phi_n\|_{n,\infty,\Omega}.$$

Trick: Normalize by defining $\tilde{\phi}_n := \dfrac{\phi_n}{n \|\phi_n\|_{n,\infty,\Omega}}$. (notice that all the ϕ_n must be nonzero, otherwise the inequality above would not hold). Then, if $j \leq n$,

$$\left\|\tilde{\phi}_n\right\|_{j,\Omega,\infty} \leq \left\|\tilde{\phi}_n\right\|_{n,\Omega,\infty} = \frac{1}{n} \to 0,$$

meaning $\tilde{\phi}_n \to 0$ in $\mathcal{D}$. However, $\left|T(\tilde{\phi}_n)\right| > 1$ for all n, meaning T is not continuous.

$\underline{\impliedby}$ The boundedness condition immediately implies continuity, since if $\phi_j \to \phi$ in $\mathcal{D}(\Omega)$, then

$$|T(\phi_j) - T(\phi)| \leq C \|\phi_j - \phi\|_{n,\infty,\Omega} \to 0,$$

where n is the order of T.

August 2022

Problem 1

(a)

$f_n \to f$ in weak-* if $f_n(x) \to f(x)$ for all $x \in X$.

(b)

If f_n converges weak-* to two functionals, those functionals must agree on all of X.

(c)

August 2024, problem 2(a).

(d)

Since $\{f_n(x)\}_{n=1}^{\infty}$ converges for all $x \in X$, the collection $\{f_n\}_{n=1}^{\infty}$ is pointwise bounded and therefore uniformly bounded by PUB.

Problem 2

(a)

Since $\mathcal{B} \subseteq Y$, we have $d(x_0, Y) \leq d(x_0, \mathcal{B})$. Also, $\|x_0\| = d(x_0, 0) \geq d(x_0, \mathcal{B})$. Since Y is a finite-dimensional subspace, it is closed, and so $d(x_0, Y) = 0$ would imply $x_0 \in Y$ and therefore $d(x_0, \mathcal{B}) = 0$ as well. Hence, we can assume $d(x_0, Y) > 0$. Then, let $0 < \varepsilon < d(x_0, Y)$. There exists $y_0 \in Y$ such that $\|x_0 - y_0\| < d(x_0, Y) + \varepsilon$. Now, if $y_0 \notin \mathcal{B}$, then $\|y_0\| > 3\|x_0\|$, meaning

$$2d(x_0, Y) > d(x_0, Y) + \varepsilon > \|x_0 - y_0\| \geq |\|x_0\| - \|y_0\|| > 2\|x_0\| \geq 2d(x_0, \mathcal{B}) \geq 2d(x_0, Y),$$

a contradiction. Thus, $y_0 \in \mathcal{B}$, meaning

$$d(y_0, \mathcal{B}) \leq \|x_0 - y_0\| < d(x_0, Y) + \varepsilon.$$

Sending $\varepsilon \to 0$ gives the other inequality.

(b)

Since $\mathcal{B}$ is a closed and bounded subset of a finite-dimensional space, it is compact by Heine-Borel. Now, consider the map $y \mapsto \|x_0 - y\|$ from $\mathcal{B}$ to $[0, \infty)$. This is continuous by triangle inequality, and so it achieves a minimum on $\mathcal{B}$. In other words, there exists $y_0 \in \mathcal{B}$ such that

$$\|x_0 - y\| = \min_{z \in \mathcal{B}} \|x_0 - z\| = \inf_{z \in \mathcal{B}} \|x_0 - z\| = d(x_0, \mathcal{B}) = d(x_0, Y).$$

(c)

Using the example in the hint, consider $x_0 = (1, -1)$. Then, for any $(a, a) \in Y = \operatorname{span}(1, 1)$, we have

$$\|(a, a) - (1, -1)\|_{\ell^1} = |a - 1| + |a + 1|.$$

The right hand side attains its minimum at multiple values, for example, $a = 0$ and $a = 1$. Hence, $(0, 0)$ and $(1, 1)$ are both best approximations of x_0 in Y.

Problem 3

(a)

If $\{x_n\}_{n=1}^\infty$ is a bounded sequence, then so is $\{Sx_n\}_{n=1}^\infty$, meaning $\{(TS)x_n\}_{n=1}^\infty$ has a convergent subsequence. Likewise, $\{Tx_n\}_{n=1}^\infty$ has a convergent subsequence $\{Tx_{n_k}\}_{k=1}^\infty$, and so $\{(ST)x_{n_k}\}_{k=1}^\infty$ converges.

(b)

Quote the spectral theorem for compact operators. In particular, the spectrum of a compact operator consists of only countably many eigenvalues (except possibly 0). Also, each nonzero eigenvalue has a finite-dimensional eigenspace. If there are infinitely many eigenvalues, then they converge to 0 and 0 is a spectral value (since the spectrum is compact). If X is infinite-dimensional, 0 is in the spectrum.

(c)

Notice that $S+T = S(I+S^{-1}T) = S(S^{-1}T)_{-1}$. By the open mapping theorem, S^{-1} is a bounded linear operator, so $S^{-1}T$ is compact by part (a). Also, $(S^{-1}T)_{-1} = S^{-1}(S+T)$, which is injective. If -1 was a spectral value of $S^{-1}T$, then it must be an eigenvalue by part (b), meaning $(S^{-1}T)_{-1}$ would not be injective. Thus, $-1 \in \rho(S^{-1}T)$, meaning $(S^{-1}T)_{-1}$ is invertible. It follows that $S+T$ is invertible.

January 2022

Problem 1

(a)

For any $f \in X^*$, $|f(x_n) - f(x)| \le \|f\| \cdot \|x_n - x\| \to 0$.

(b)

Suppose x and y are both weak limits of $\{x_n\}_{n=1}^{\infty}$. Then, $f(x) = f(y)$ for all $f \in X^*$. By Hahn-Banach, there exists $g \in X^*$ such that $\|g\| = 1$ and $g(x-y) = \|x-y\|$. The left hand side is zero, so $x = y$.

(c)

Consider the collection of evaluation maps $\{E_{x_n}\}_{n=1}^{\infty} \subseteq X^{**}$. This collection is pointwise bounded since $\{E_{x_n}(f)\}_{n=1}^{\infty} = \{f(x_n)\}_{n=1}^{\infty}$ converges for all $f \in X^*$. By PUB, $\{E_{x_n}\}_{n=1}^{\infty}$ is bounded, but $\|E_y\| = \|y\|$ for all $y \in X$, meaning $\{x_n\}_{n=1}^{\infty}$ is bounded too.

(d)

Take $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. We have

$$\|x\| = f(x) = \lim_{n\to\infty} f(x_n) \le \liminf_{n\to\infty} \|f\| \, \|x_n\| = \liminf_{n\to\infty} \|x_n\| .$$

Problem 2

(a)

Let (X, d) be a complete metric space and $\{U_n\}_{n=1}^\infty$ a countable collection of open, dense subsets of X. Then, $\bigcap_{n=1}^\infty U_n$ is dense in X as well.

Important consequence: X is not a countable union of nowhere dense sets.

(b)

If $\{e_n\}_{n=1}^\infty$ was a Hamel basis, then $X = \bigcup_{n=1}^\infty \operatorname{span}\{e_1, \dots, e_n\}$. Fix $n \in \mathbb{N}$; we will show that $Y := \operatorname{span}\{e_1, \dots, e_n\}$ is nowhere dense. Since finite dimensional subspaces are always closed, this amounts to showing $\operatorname{span}\{e_1, \dots, e_n\}$ has empty interior. Suppose Y contains an open ball $B_\delta(x)$. Then, since Y is closed, it must contain $\overline{B_\delta(x)}$ as well. However, this would suggest that $\overline{B_\delta(x)}$ is compact by Heine-Borel being a closed and bounded subset of a finite-dimensional subspace. This is a contradiction since X is infinite-dimensional, so Y contains no open ball, i.e., its interior is empty. Because of this, $X \neq \bigcup_{n=1}^\infty \operatorname{span}\{e_1, \dots, e_n\}$, so there is no countably infinite Hamel basis.

Problem 3

(a)

See " $\Longleftarrow$ " on problem 4 of January 2024.

(b)

Compute

$$\langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\langle Tx, x \rangle}.$$

(c)

This proof consists of three parts:

1. Show the *point* spectrum is real.
2. Show that $\lambda \in \rho(T)$ if T_λ is bounded below.
3. Show that the whole spectrum is real.

First, let $\lambda \in \sigma_p(T)$ with x a corresponding eigenvector. Then,

$$\langle Tx, x \rangle = \lambda \langle x, x \rangle \implies \lambda = \frac{\langle Tx, x \rangle}{\|x\|^2} \in \mathbb{R}$$

by part (b).

Next, if T_λ is bounded below, then it is injective and has a closed range by part (a). We claim $R(T_\lambda) = H$. If this is not the case, then, since $R(T_\lambda)$ is closed, there exists some nonzero $x \in (R(T_\lambda))^\perp$. Thus,

$$\langle T_{\bar{\lambda}}x, y\rangle = \langle Tx, y\rangle - \bar{\lambda}\langle x, y\rangle = \langle x, Ty\rangle - \langle x, \lambda y\rangle = \langle x, T_\lambda y\rangle = 0$$

for all $y \in H$, meaning $T_{\bar{\lambda}}x = 0$. But this suggests $\bar{\lambda} \in \sigma_p(T)$, and since the eigenvalues of T are real, $\lambda \in \sigma_p(T)$. This is a contradiction since T_λ is injective, so $R(T_\lambda) = H$. Finally, T_λ being a bijection also means its inverse is bounded by open mapping theorem, so $\lambda \in \rho(T)$.

Finally, suppose $\lambda := \alpha + i\beta \in \sigma(T)$. Then, for all x,

$$\langle T_\lambda x, x\rangle - \overline{\langle T_\lambda x, x\rangle} = \langle Tx, x\rangle - \lambda \|x\|^2 - \langle Tx, x\rangle + \overline{\lambda}\|x\|^2 = -2i\beta\|x\|^2 .$$

If $\beta \neq 0$, then by triangle inequality and Cauchy-Schwarz,

$$|\beta|\,\|x\|^2 = \frac{1}{2}\left[|\langle T_\lambda x, x\rangle| - \left|\overline{\langle T_\lambda x, x\rangle}\right|\right] \le |\langle T_\lambda x, x\rangle| \le \|T_\lambda x\|\,\|x\|,$$

meaning $\|T_\lambda x\| \ge |\beta|\,\|x\|$, i.e., T_λ is bounded below. However, this means $\lambda \in \rho(T)$, a contradiction. So, we must have $\beta = 0$, i.e., λ is real.

August 2021

Problem 1

(a)

Clearly, f_y is linear, and the sum defining $f_y(x)$ is finite for all $x \in c_0$ since

$$|f_y(x)| \le \sum_{j=1}^{\infty} |x_j \overline{y_j}| \le \|x\|_{\ell^\infty} \|y\|_{\ell^1} .$$

Also, this suggests $\|f_y\| \le \|y\|_{\ell^1}$.

(b)

Letting (x^n) be the sequences given in the hint, we find that $\|(x^n)\|_{\ell^\infty} \le 1$ for all n, and

$$\|f_y\| \ge |f_y(x^n)| = \left|\sum_{j=1}^{\infty} x_j^n \overline{y}_j\right| = \sum_{j=1}^{n-1} |y_j| .$$

Since $y \in \ell^1$, $|f_y(x^n)| \to \|y\|_{\ell^1}$ as $n \to \infty$, giving us $\|f_y\| \ge \|y\|_{\ell^1}$.

(c)

Let $e^{(1)}, e^{(2)}, \ldots,$ be the usual Schauder basis of c_0, and let y be the sequence defined by $y_j = \overline{f(e^{(j)})}$.

We now claim $y \in \ell^1$. First, for all $N \in \mathbb{N}$, define $x^{(N)}$ by

$$x_j^{(N)} := \begin{cases} \operatorname{sgn}(\overline{y_j}) & j \le N, \\ 0 & j > N \end{cases},$$

where $\operatorname{sgn}(z)$ is the sign function on the complex numbers. Each $x^{(N)}$ is in c_0 since they are eventually 0, and $\left\|x^{(N)}\right\| \le 1$ for all N. Hence,

$$\sum_{j=1}^{N} |y_j| = \sum_{j=1}^{\infty} x_j^{(N)} \overline{y_j} = \sum_{j=1}^{\infty} x_j^{(N)} f(e^{(j)}) = f(x^{(N)}) \le \|f\| \left\|x^{(N)}\right\|_\infty \le \|f\|.$$

Since $\|f\|$ is fixed and finite, we can send $N \to \infty$ to get $\sum_{j=1}^{\infty} |y_j| < \infty$.

Now, for all $x \in c_0$, $\sum_{j=1}^{n} x_j e^{(j)} \to x$ in ℓ^∞ since $x_j \to 0$. This gives us

$$f(x) = \lim_{n\to\infty} f\left(\sum_{j=1}^{n} x_j e^{(j)}\right) = \lim_{n\to\infty} \sum_{j=1}^{n} x_j \overline{y_j} = f_y(x).$$

Problem 2

(a)

Recall that $\inf_{y\in M} \|x - y\| = \|x - P_M x\|$, where P_M is the orthogonal projection onto M. Also, since $P_M x \in M$, we have $PP_M x = P_M x$, giving us

$$\begin{aligned} \|x - Px\| &\le \|x - P_M x\| + \|P_M x - Px\| = \|x - P_M x\| + \|P(P_M x - x)\| \\ &\le (\|P\| + 1)\|x - P_M x\| = (\|P\| + 1) \inf_{y\in M} \|x - y\|, \end{aligned}$$

and so take $C := \|P\| + 1$.

(b)

$\underline{\Longrightarrow}$ If P is an orthogonal projection, then $N = M^\perp$. So,

$$\inf_{y\in N, \|y\|=1, x\in M} \|y - x\|^2 = \inf_{y\in N, \|y\|=1, x\in M} \left(\|y\|^2 + \|x\|^2\right) = \inf_{x\in M} \left(1 + \|x\|^2\right) = 1.$$

Thus, $\inf_{y\in N, \|y\|=1, x\in M} \|y - x\| = 1$ as well.

$\underline{\Longleftarrow}$ We claim that for all $z \in H$, $z - Pz = Qz \perp M$. To this end, fix

$z \in H$ and define $y = Qz / \|Qz\|$ (of course, if $Qz = 0$, then it is orthogonal to M, so we can assume it is not zero). Then, $y \in N$ with $\|y\| = 1$, and so

$$1 = \inf_{w \in N, \|w\|=1, x \in M} \|x - w\| \leq \inf_{x \in M} \|x - y\| \leq \|y\| = 1.$$

Thus, the element in M closest to y is 0, i.e., $P_M y = 0$. This means $y \in M^\perp$, so $z - Pz \in M^\perp$ as well.

From here, P is the orthogonal projection onto M, since for any $z \in H$ and $y \in M$,

$$\|z - y\|^2 = \|z - Pz + Pz - y\|^2 = \|z - Pz\|^2 + \|Pz - y\|^2 \geq \|z - Pz\|^2$$

since $Pz - y \in M$ and is therefore orthogonal to $z - Pz$.

Problem 3

(a)

State PUB

(b)

Use triangle inequality:

$$\|L_n x_n - Lx\| \leq \|L_n x_n - L_n x\| + \|L_n x - Lx\| \leq \|L_n\| \|x_n - x\| + \|L_n x - Lx\|.$$

Since $\{L_n x\}_{n=1}^\infty$ converges for all $x \in X$, $\{L_n\}_{n=1}^\infty$ is bounded by PUB. So, the first term on the RHS converges to 0. Likewise, the second term converges to 0 since $L_n \to L$ in weak-*.

(c)

Using the example, we have that $e_n \rightharpoonup 0$ since $(\ell^2)^* = \ell^2$. From here, let L_n be given by $L_n x = x_n$. Then, $L_n \to 0$ in weak-*, but $L_n(x_n) = 1$ for all n.

January 2021

Problem 1

Use closed graph theorem. Let $(x_n, Ax_n) \to (x, y)$. Then, $x_n \to x$ and $Ax_n \to y$, so $BAx_n \to By$ since B is continuous. But now, $(x_n, BAx_n) \to (x, By)$, and since BA is continuous, $By = BAx$ by closed graph theorem. But now, B is injective, so $y = Ax$.

Problem 2

$\underline{\implies}$ Immediate from PUB

$\underline{\impliedby}$ Let $x \in X$, $\varepsilon > 0$, and y an element of this dense subset such that $\|x - y\| < \varepsilon$. Then,

$$\begin{aligned}\|A_n x - A_m x\| &\leq \|A_n x - A_n y\| + \|A_n y - A_m y\| + \|A_m y - A_m x\| \\ &\leq \varepsilon(\|A_n\| + \|A_m\|) + \|A_n y - A_m y\|.\end{aligned}$$

Since $\{A_n y\}_{n=1}^{\infty}$ is Cauchy, the second term goes to 0, and from the uniform bound, the first term can also be made arbitrarily small.

Problem 3

Recall that precompact sets are separable. Now,

$$Y = A(X) = A\left(\bigcup_{n=1}^{\infty} B_n(0)\right) = \bigcup_{n=1}^{\infty} A(B_n(0)).$$

Since A^{-1} is compact, each $A(B_n(0))$ is precompact and therefore separable. This means Y is also separable (just join the countable dense subsets of each member of the union).

For the second part, we can relate the spectra of A and A^{-1}. By hypothesis, $0 \in \rho(A)$. Notice that for any $\lambda \neq 0$,

$$A - \lambda I = -\lambda A(A^{-1} - \lambda^{-1} I).$$

But A is injective, has a dense range, and has a bounded inverse, so if $\lambda \in \sigma(A)$, then $\lambda^{-1} \in \sigma(A^{-1})$. But A^{-1} is compact, so its nonzero spectral values are all eigenvalues. This then forces $\lambda \in \sigma_p(A)$, for if $A^{-1}x = \lambda^{-1}x$, then $Ax = \lambda x$. So, all spectral values of A are eigenvalues.

Problem 4

Don't worry about this one; Fourier transform is not an Applied I topic.

August 2020

Problem 1

By PUB, $M := \sup_{s \in S} \|T_s\| < \infty$. Let X_σ be the set of $x \in X$ such that $T_s x \to T_\sigma x$ as $s \to \sigma$. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in X_σ converging to x, and let $\{s_m\}_{m=1}^\infty$ be a sequence in S converging to σ. We claim $T_{s_n}x \to T_\sigma x$ as $n \to \infty$. Use triangle inequality:

$$\begin{aligned}\|T_{s_m}x - T_\sigma x\| &\le \|T_{s_m}x - T_{s_m}x_n\| + \|T_{s_m}x_n - T_\sigma x_n\| + \|T_\sigma x_n - T_\sigma x\| \\ &\le 2M\,\|x - x_n\| + \|T_{s_m}x_n - T_\sigma x_n\|\,.\end{aligned}$$

We now have two indices, so some care should be taken. First, let $\varepsilon > 0$ and pick $n \in \mathbb{N}$ such that

$$\|x - x_n\| < \frac{\varepsilon}{4(M+1)}.$$

Then, since $x_n \in X_\sigma$, there exists $M \in \mathbb{N}$ such that

$$\|T_{s_m}x_n - T_\sigma x_n\| < \frac{\varepsilon}{2}$$

for all $m \ge M$. With this setup, we will have $\|T_{s_m}x - T_\sigma x\| < \varepsilon$ for $m \ge M$.

Problem 2

The weak closure of the unit sphere is the unit ball:

https://math.stackexchange.com/questions/153889/prove-the-weak-closure-of-the-unit-sphere-is-the-unit-ball.

In particular, 0 is in the weak closure of the unit sphere, so there exists a sequence in the unit sphere converging weakly to 0.

Problem 3

$\underline{\implies}$ I believe we need to assume Y is separable as well in order to prove this. If we assume this much, we have a *countable* ON basis $\{e_n\}_{n=1}^\infty$, and by Riesz-Fischer,

$$Tx = \sum_{n=1}^\infty \langle Tx, e_n\rangle\, e_n.$$

The partial sums are finite-rank operators that converge in operator norm to T.

$\underline{\impliedby}$ Same as January 2025, problem 2 since finite-rank operators are compact by Heine-Borel

Problem 4

Same as August 2023, problem 2.

January 2020

Problem 1

(a)

State CGT.

(b)

Suppose $(x_n, Ax_n) \to (x, y)$. We need to show $y = Ax$ to apply the closed graph theorem. To this end, let $z \in X$ and compute

$$\langle y, z\rangle = \lim_{n\to\infty} \langle Ax_n, z\rangle = \lim_{n\to\infty} \langle Az, x_n\rangle = \langle Az, x\rangle = \langle Ax, z\rangle \,,$$

and so $Ax = y$.

(c)

https://math.stackexchange.com/questions/216858/positive-operator-is-bounded

Problem 2

(a)

By inspection, there is no convergence at $x = 0$. For $x \neq 0$, think of the limit as a continuous one and use L'Hôpital's rule:

$$\frac{n^{1/p-1}}{pxe^{nx}} = \frac{1}{pxn^{(p-1)/p}e^{nx}} \to 0$$

since $\dfrac{p-1}{p} \geq 0$.

(b)

Just compute the L^p norm:

$$\int_0^1 |g_n(x)|^p = \int_0^1 ne^{-pnx}\,dx = -\frac{n}{pn}e^{-pnx}\Big|_0^1 = \frac{1}{p}\left[1 - e^{-pn}\right] \to \frac{1}{p}.$$

In particular, $g_n \not\to 0$ in L^p.

(c)

Recall that $g_n \rightharpoonup 0$ weakly in L^p if

$$\int_0^1 fg_n \to 0$$

for all $f \in L^q$, where q is the conjugate exponent of p. This does not work when $p = 1$, since in this case, we can take $f = \chi_{[0,1]}$, and by part (b), $\int_0^1 g_n \to 1$. Now, suppose $p > 1$. First, notice that for $0 \le a < b \le 1$,

$$\int_a^b g_n(x)\,dx = n^{(1-p)/p}[e^{-an} - e^{-bn}] \le 2n^{(1-p)/p} \to 0$$

since $\dfrac{1-p}{p} < 0$. So, for all step functions f, $\int f g_n \to 0$. Since step functions are dense in L^q, we have weak convergence.

Problem 3

Let $Z = Y + \mathbb{F}w$, which is a linear subspace of X. Then, define $f : Z \to \mathbb{F}$ as $f(y + \lambda w) = \lambda d$. Clearly, f is linear with $f(w) = d$ and $f(y) = 0$ for all $z \in Y$. We will check that $\|f\| \le 1$. Indeed, if $y + \lambda w \in Z$ with $\lambda \ne 0$, then

$$|f(y + \lambda w)| = |\lambda|\, d = \frac{\|y + \lambda w\|}{\|y + \lambda w\|}\, |\lambda|\, d = \frac{\|y + \lambda w\|}{\|\lambda^{-1} y + w\|} d \le \|y + \lambda w\|$$

since $\lambda^{-1} y + w = w - (-\lambda^{-1} y)$, which is a distance between w and Y and therefore greater than d. The inequality $|f(y + \lambda w)| \le \|y + \lambda w\|$ still holds when $\lambda = 0$, so $\|f\| \le 1$. From here, use Hahn-Banach to extend to all of X; this extension will still have all the desired properties.

August 2019

Problem 1

(a)

Quote the theorem: If A is a self-adjoint, compact operator, then there exists an orthonormal basis of eigenvectors $\{u_\alpha\}$ with eigenvalues $\{\lambda_\alpha\}$ such that for all $x \in H$,

$$Ax = \sum_{\alpha \in \mathcal{I}} \lambda_\alpha \langle x, u_\alpha \rangle u_\alpha.$$

(b)

Using Arzelà–Ascoli and the density of $C(\Omega)$ in $L^2(\Omega)$, T is compact. Also, since K is symmetric, T is also self-adjoint. So, we can apply the spectral theorem above, and since $L^2(\Omega)$ is separable, we can assume this ON basis of eigenvectors is countable. Call this ON basis $\{e_j\}_{j=1}^\infty$ and their eigenvalues $\{\lambda_j\}_{j=1}^\infty$. Notice that

$$\lambda_j e_j(x) = T e_j(x) = \int K(x, y) e_j(y)\, dy,$$

and since K is nonnegative, the eigenvalues λ_j must also be nonnegative. Now, for all $f \in L^2$ and $x \in \Omega$,

$$\begin{aligned}
\int_\Omega \left[\sum_{j=1}^{\infty} \lambda_j e_j(x) e_j(y)\right] f(y)\, dy &= \sum_{j=1}^{\infty} \lambda_j e_j(x) \int_\Omega f(y) e_j(y)\, dy \\
&= \sum_{j=1}^{\infty} \lambda_j \langle f, e_j \rangle e_j(x) && \text{the field is real} \\
&= Tf(x) \\
&= \int_\Omega K(x,y) f(y)\, dy.
\end{aligned}$$

On the first line, we use DCT to exchange integration with summation. Since this holds for all $f \in L^2$, we must have $K(x,y) = \sum_{j=1}^{\infty} \lambda_j e_j(x) e_j(y)$.

(c)

Directly compute

$$\int_\Omega K(x,x)\, dx = \int_\Omega \sum_{j=1}^{\infty} \lambda_j e_j^2(x)\, dx = \sum_{j=1}^{\infty} \lambda_j \int_\Omega e_j^2(x)\, dx = \sum_{j=1}^{\infty} \lambda_j,$$

where monotone convergence theorem lets us swap integration with summation (notice that each term is nonnegative).

Problem 2

(a)

Directly compute:

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= 2\|x\|^2 + 2\|y\|^2 + \langle x, y \rangle + \langle y, x \rangle - \langle x, y \rangle - \langle y, x \rangle \\
&= 2(\|x\|^2 + \|y\|^2).
\end{aligned}$$

(b)

First, we need to reveal a candidate for the best approximation. Let $\delta = \operatorname{dist}(x, M)$. Then, there is a sequence $\{y_n\}_{n=1}^{\infty}$ in M such that $\delta_n := \|x - y_n\| \to$

δ. We claim that $\{y_n\}_{n=1}^{\infty}$ is Cauchy. For this, we use the parallelogram law:

$$\begin{aligned}
\|y_n - y_m\|^2 &= \|(y_n - x) + (x - y_m)\|^2 \\
&= 2(\|y_n - x\|^2 + \|x - y_m\|^2) - \|(y_n - x) - (x - y_m)\|^2 \\
&= 2(\delta_n^2 + \delta_m^2) - 4\left\|\frac{y_n + y_m}{2} - x\right\|^2 \\
&\leq 2(\delta_n^2 + \delta_m^2) - 4\delta^2 \\
&\to 4\delta^2 - 4\delta^2 \\
&= 0,
\end{aligned}$$

where on the fourth line, we use the fact that $\dfrac{y_n + y_m}{2} \in M$ due to convexity. Since $\{y_n\}_{n=1}^{\infty}$ is Cauchy, it has a limit y, and the closedness of M means $y \in M$. Then, y is the best approximation of x in M, since, by the continuity of $\|\cdot\|$,

$$\|x - y\| = \lim_{n\to\infty} \|x - y_n\| = \lim_{n\to\infty} \delta_n = \delta = \inf_{z\in M} \|x - z\|.$$

For uniqueness, suppose $\|x - z\| = \delta$ for some $z \in M$. Then, apply parallelogram law again:

$$\begin{aligned}
\|y - z\|^2 &= \|(y - x) + (x - z)\|^2 \\
&= 2(\|y - x\|^2 + \|x - z\|^2) - \|(y - x) - (x - z)\|^2 \\
&= 4\delta^2 - 4\left\|\frac{y + z}{2} - x\right\|^2 \\
&\leq 4\delta^2 - 4\delta^2 \\
&= 0,
\end{aligned}$$

meaning $y = z$.

Problem 3

(a)

Dual spaces are complete, so we can use closed graph theorem. Suppose $(f_n, Sf_n) \to (f, g)$. Then, since $f_n \to f$ strongly, $f_n \to f$ in weak-* as well. This means $Sf_n \to Sf$ in weak-* by hypothesis. However, $Sf_n \to g$ strongly and therefore in weak-*, and since weak-* limits are unique, $g = Sf$, meaning Graph(S) is closed.

(b)

If $f_n \to f$ in weak-*, then for all $x \in X$,

$$(T^* f_n)(x) = f_n(Tx) \to f(Tx) = (T^* f)(x),$$

so $T^* f_n \to T^* f$ in weak-*, meaning T^* is weakly-* sequentially continuous.

(c)

Follows from (a) and (b).

January 2019

Problem 1

(a)

$\underline{\Longrightarrow}$ Easy:

$$\lambda \langle x, x \rangle = \langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\lambda} \langle x, x \rangle .$$

$\underline{\Longleftarrow}$ Fix $x, y \in H$. Then, apply T to $x - y$:

$$\langle T(x-y), x-y \rangle = \langle Tx, x \rangle + \langle Ty, y \rangle - \langle Tx, y \rangle - \overline{\langle x, Ty \rangle}.$$

The left hand side and the first two terms on the right hand side are real, so $\langle Tx, y \rangle + \overline{\langle x, Ty \rangle} \in \mathbb{R}$; this means the imaginary parts of $\langle Tx, y \rangle$ and $\langle x, Ty \rangle$ agree.

To see that their real parts agree, this time, apply T to $ix - y$:

$$\langle T(ix-y), ix-y \rangle = -\langle Tx, x \rangle + \langle Ty, y \rangle - i \langle Tx, y \rangle + i\overline{\langle x, Ty \rangle}.$$

Now, $\langle Tx, y \rangle - \overline{\langle x, Ty \rangle}$ is a purely imaginary number, so the real parts of $\langle Tx, y \rangle$ and $\langle x, Ty \rangle$ must agree. Hence, $\langle Tx, y \rangle = \langle x, Ty \rangle$, so T is self-adjoint.

(b)

Same as August 2019, problem 1(a)

(c)

From the spectral theorem,

$$Tx = \sum_{\alpha \in \mathcal{I}} \lambda_\alpha \langle x, e_\alpha \rangle e_\alpha.$$

From here, we can compute $\langle Tx, x \rangle$ for some inspiration on how to define P and N:

$$\langle Tx, x \rangle = \sum_{\alpha \in \mathcal{I}} \langle \lambda_\alpha \langle x, e_\alpha \rangle e_\alpha, x \rangle = \sum_{\alpha \in \mathcal{I}} \lambda_\alpha \langle x, e_\alpha \rangle \langle e_\alpha, x \rangle = \sum_{\alpha \in \mathcal{I}} \lambda_\alpha \left| \langle x, e_\alpha \rangle \right|^2$$

(remember that the inner product is continuous and the sum above must be a countable sum by Riesz-Fischer, so exchanging the inner product and sum is legit). In particular, to guarantee that T is a positive operator, we should pick out all the positive values of λ_α. This leads us to define $\beta_\alpha = \max(\lambda_\alpha, 0)$ and $\gamma_\alpha = \max(-\lambda_\alpha, 0)$. Then, P and N are given by

$$Px = \sum_{\alpha \in \mathcal{I}} \beta_\alpha \langle x, e_\alpha \rangle e_\alpha,$$
$$Nx = \sum_{\alpha \in \mathcal{I}} \gamma_\alpha \langle x, e_\alpha \rangle e_\alpha.$$

With this, it is easy to check that P and N have all the desired properties.

Problem 2

(a)

Same as January 2022, problem 1(d)

(b)

The closed unit ball in X^* is compact in the weak-* topology of X^*.

(c)

Bounded below $\implies$ one-to-one, so uniqueness is guaranteed. Thus, we only need to show existence, and for this, the Banach-Alaoglu theorem will surely be used.

Let $f \in Y$ and let $\{f_n\}_{n=1}^\infty$ be a sequence in D converging to f. For all n, there exists x_n such that $Tx_n = f_n$. Since T is continuous, we are lead to believe that a solution x to $Tx = f$ might be obtained by taking a limit of $\{x_n\}_{n=1}^\infty$ in some way. Notice that $M := \sup_n \|f_n\| < \infty$ since $\{f_n\}_{n=1}^\infty$ converges, and since T is bounded below,

$$\|x_n\| \leq \gamma \|Tx_n\| = \gamma \|f_n\| \leq \gamma M$$

for some $\gamma > 0$. In other words, the sequence $\{x_n\}_{n=1}^\infty$ is bounded. By combining Banach-Alaoglu with the fact that X is symmetric and reflexive, $\{x_n\}_{n=1}^\infty$ has a weakly convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$; let x be the weak limit of this subsequence. From here, we claim $Tx = f$.

Since $x_{n_k} \rightharpoonup x$ and T is continuous, $Tx_{n_k} \rightharpoonup Tx$, meaning

$$h(Tx) = \lim_{k\to\infty} h(Tx_{n_k}) = \lim_{k\to\infty} h(f_{n_k}) = h(f)$$

for all $h \in Y^*$; this is enough to conclude that $Tx = f$.

Problem 3

(a)

$\underline{\Longrightarrow}$ By Cauchy-Schwartz,

$$|L_y(f)| = |f(y)| = |\langle f, K(\cdot, y)\rangle| \le \|f\| \, \|K(\cdot, y)\| ,$$

so $\|L_y\| \le \|K(\cdot, y)\| < \infty$.

$\underline{\Longleftarrow}$ For all $y \in \mathbb{R}^d$, Riesz representation gives some $g_y \in H$ such that $L_y(f) = \langle f, g_y\rangle$ for all $f \in H$. Define $K(x, y) := g_y(x)$. Then, $K(\cdot, y) = g_y \in H$, and $\langle f, K(\cdot, y)\rangle = \langle f, g_y\rangle = L_y(f)$ for all f.

(b)

Suppose K and K' are both reproducing kernel functions. Then, $\langle f, K(\cdot, y)\rangle = \langle f, K'(\cdot, y)\rangle$ for all $y \in \mathbb{R}^d$ and $f \in H$. This means $K(\cdot, y) = K'(\cdot, y)$ for all $y \in \mathbb{R}^d$, and so $K = K'$.

For the other result, compute

$$K(x, y) = \langle K(\cdot, y), K(\cdot, x)\rangle = \overline{\langle K(\cdot, x), K(\cdot, y)\rangle} = \overline{K(y, x)}.$$

(c)

By Riesz representation, there exists some $g \in H$ such that $Lf = \langle f, g\rangle$ for all $f \in H$. Thus,

$$z(y) = \overline{LK(\cdot, y)} = \overline{\langle K(\cdot, y), g\rangle} = \langle g, K(\cdot, y)\rangle = g(y).$$

Thus, $z = g \in H$. Moreover, $Lz = \langle z, z\rangle = \|z\|^2$, and again by Riesz, $\|z\|^2 = \|L\|^2$.

January 2017

Problem 1

(a)

Since P_j is an orthogonal projection, $M_j = N_j^\perp$. This gives us

$$\|x\|^2 = \left\|P_j x + P_j^\perp x\right\|^2 = \|P_j x\|^2 + \left\|P_j^\perp x\right\|^2 \ge \|P_j x\|^2 .$$

Thus, $\|P_j\| \le 1$. For positivity,

$$\langle P_j x, x\rangle = \left\langle P_j x, P_j x + P_j^\perp x\right\rangle = \langle P_j x, P_j x\rangle = \|P_j x\| \ge 0$$

since $P_j x \perp P_j^\perp x$.

(b)

Follow the order given in the hint:

<u>(i) $\implies$ (ii)</u> $\|P_1 x\| = \|P_1 P_2 x\| \le \|P_1\| \|P_2 x\| \le \|P_2 x\|$ by part (a).

<u>(ii) $\implies$ (iii)</u> Use the computation at the end of part (a):

$$\langle (P_2 - P_1)x, x\rangle = \langle P_2 x, x\rangle - \langle P_1 x, x\rangle = \|P_2 x\| - \|P_1 x\| \ge 0,$$

and so $P_2 \ge P_1$.

<u>(iii) $\implies$ (iv)</u> Suppose $x \in N_2$. Then,

$$0 \le \langle (P_2 - P_1)x, x\rangle = -\langle P_1 x, x\rangle = -\|P_1 x\| \le 0.$$

The only way this can hold is if $P_1 x = 0$, i.e., if $x \in N_1$.

<u>(iv) $\implies$ (v)</u> Recall that $M_j = N_j^{\perp}$, and from the definitions, it is clear that $M_1 \subseteq M_2$ if $N_1 \supseteq N_2$.

<u>(v) $\implies$ (i)</u> Let $x \in H$. Then, $P_1 x \in M_1 \subseteq M_2$, so $P_2(P_1 x) = P_1 x$, meaning $P_2 P_1 = P_1$. Likewise,

$$(P_1 P_2)x = P_1(x - P_2^{\perp} x) = P_1 x - P_1 P_2^{\perp} x.$$

Now, $P_2^{\perp} x \in N_2 \subseteq N_1$ (remember that $\perp$ reverses set inclusions), so $P_1 P_2^{\perp} x = 0$, establishing $P_1 P_2 = P_1$.

Problem 2

(a)

Technically, B^* maps X^{**} to Y^*. By $B^* x$, they mean $B^*(E_x)$, where E_x is the evaluation map in X^{**} associated with x.

(b)

We are told that A is onto, and if $Ax = 0$, then $\|x\|^2 \le \alpha^{-1} Ax(x) = 0$, meaning A is injective as well. Thus, A is invertible with a bounded inverse A^{-1} by OMT. Now, for all nonzero $y \in X$,

$$\|y\|^2 \le \alpha^{-1} |Ay(y)| \le \alpha^{-1} \|Ay\| \|y\|,$$

and so $\|y\| \le \alpha^{-1} \|Ay\|$. Replacing y with $A^{-1}x$ (for x nonzero) gives

$$\left\|A^{-1} x\right\| \le \alpha^{-1} \|x\|,$$

and so $\left\|A^{-1}\right\| \le \alpha^{-1}$.

(c)

Suppose (x', y') is also a solution and set $w = x - x'$ and $z = y - y'$. Since A, B, and C are linear, we see that (w, z) is a solution to

$$\begin{aligned} Aw - Bz &= 0, \\ B^* w + Cz &= 0. \end{aligned}$$

From these equations, we have $Aw(w) - Bz(w) = 0$ and $B^* w(z) + Cz(z) = 0$. Since $B^* w(z) = E_w(Bz) = Bz(w)$, we have $Aw(w) + Cz(z) = 0$ by adding the previous equations. Thus,

$$0 = Aw(w) + Cz(z) \geq \alpha \left\| w \right\|^2 + \gamma \left\| z \right\|^2,$$

which forces $w = z = 0$, i.e., $x = x'$ and $y = y'$.

(d)

Not sure. If you have a solution to this part, please reach out!

Problem 3

(a)

Our kernel is $\sin((x + y)/2)$, which is in $L^2(I^2)$. So, compactness follows from density and Arzelà–Ascoli. For self-adjointness, use Fubini's theorem:

$$\begin{aligned} \langle Af, g \rangle = \int_0^1 Af(x)\overline{g(x)}\, dx &= \int_0^1 \int_0^1 f(y) \sin\left(\frac{x+y}{2}\right) \overline{g(x)}\, dy dx \\ &= \int_0^1 f(y) \overline{\left[\int_0^1 \sin\left(\frac{x+y}{2}\right) g(x)\, dx\right]} dy \\ &= \int_0^1 f(y)\overline{Ag(y)}\, dy \\ &= \langle f, Ag \rangle . \end{aligned}$$

(b)

Denote $K(x, y) = \sin((x + y)/2)$. By combining Tonelli's theorem with Hölder's inequality, you will find that $\|A\| = \|K\|_2$. So, directly compute the L^2 norm of K:

$$\begin{aligned}
\|K\|_2^2 &= \int_0^1 \int_0^1 \sin^2\left(\frac{x+y}{2}\right)\, dxdy \\
&= \int_0^1 \int_0^1 \left[\frac{1-\cos(x+y)}{2}\right]\, dxdy \\
&= \frac{1}{2} - \frac{1}{2}\int_0^1 \int_0^1 \cos(x+y)\, dxdy \\
&= \frac{1}{2} - \frac{1}{2}\int_0^1 [\sin(1+y) - \sin(y)]\, dy \\
&= \frac{1}{2} - \frac{1}{2}[2\cos(1) - \cos(2) - 1] \\
&= 1 - \cos(1) - \frac{\cos(2)}{2} \\
&< 1.
\end{aligned}$$

Thus, $\|A\| < 1$.

(c)

From ST(sa), we know that $\inf_{\|f\|_2=1} \langle Af, f\rangle$ is the smallest spectral value of A, and since A is also compact, this spectral value will be an eigenvalue so long as it is nonzero. Thus, it suffices to show that there exists $f \in L^2(I)$ such that $\|f\| = 1$ and $\langle Af, f\rangle < 0$. If f is real-valued, then

$$\langle Af, f\rangle = \int_0^1 f(x) \int_0^1 f(y) \sin\left(\frac{x+y}{2}\right)\, dydx.$$

To make this integral negative, take $f(x) = 12\left(\frac{1}{2} - x\right)$ (shoutout to Jeffrey Cheng for this example). One can directly compute that $\|f\|_2 = 1$ and $\langle Af, f\rangle < 0$.

August 2016

Problem 1

Same as January 2020, problem 3

Problem 2

(a)

A W-weak open set around 0 is a union of sets of the form

$$\bigcap_{i=1}^{n} w_i^{-1}(B_{\varepsilon_i}(0))$$

for $\varepsilon_i > 0$ and $w_i \in W$.

(b)

$L^{-1}(B_1(0))$ is open in W and nonempty since it contains 0, so it must contain a W-open set around 0. That is, it contains a set of the form described in part (a). Now, if $w_1(x) = \ldots = w_n(x) = 0$, then $x \in \bigcap_{i=1}^{n} w_i^{-1}(B_{\varepsilon \cdot \varepsilon_i}(0))$ for all $\varepsilon > 0$, meaning $x \in \varepsilon \cdot L^{-1}(B_1(0)) = L^{-1}(B_\varepsilon(0))$. By sending $\varepsilon \to 0$, we see that $Lx = 0$ as well, so L is a linear combination of the w_i by the hint. This immediately tells us $L \in W$.

(c)

Notice that the collection of evaluation maps are a vector space of linear functionals that separate points of X^* and generate the weak-* topology on X^* (that is, the weak-* topology on X^* is the smallest topology on which all evaluation maps are continuous). By part (b), all weak-* continuous linear functionals on X^* are evaluation maps.

Problem 3

(a)

Suppose $\varphi_j \to \varphi$ in $\mathcal{D}$, meaning there exists $K \subseteq\subseteq \Omega$ such that $\operatorname{supp} \varphi_j \subseteq K$ for all j, and for all fixed $n \in \mathbb{N}$, $\|\varphi_j - \varphi\|_{n,\infty,\Omega} \to 0$. We claim $T\varphi_j \to T\varphi$ in $\mathcal{D}(-1,1)$. First, notice that $\operatorname{supp} T\varphi_j \subseteq \pi_1(K)$, which is compact since π_1 is continuous and K is compact. Likewise, we have

$$\begin{aligned}\left\|(T\varphi_j)^{(n)} - (T\varphi)^{(n)}\right\|_\infty &= \sup_{x \in (-1,1)} \left|\frac{\partial^n}{\partial x^n}\varphi_j(x,0) - \frac{\partial^n}{\partial x^n}\varphi(x,0)\right| \\ &\le \|\varphi_j - \varphi\|_{n,\infty,\Omega} \\ &\to 0.\end{aligned}$$

Since this holds for every derivative, we have $\|T\varphi_j - T\varphi\|_{n,\infty,\Omega} \to 0$.

(c)

For $\varphi \in \mathcal{D}(\Omega)$, we have

$$\langle T^*(\delta_0), \varphi\rangle = \langle \delta_0, T\varphi\rangle = \langle \delta_0, \varphi(\cdot, 0)\rangle = \varphi(0,0),$$

meaning $T^*(\delta_0) = \delta_{(0,0)}$.

Likewise,

$$\langle T^*(\delta_0'), \varphi\rangle = -\langle \delta_0, (T\varphi)'\rangle = -\frac{\partial \varphi}{\partial x}(0,0),$$

and so $T^*(\delta_0') = -\partial_x \delta_{(0,0)}$.

January 2016

Problem 1

(a)

Since T is positive, it is also monotone in the sense that if $f \le g$ pointwise on A, then $Tf \le Tg$ pointwise on B. In particular, every $f \in C(A)$ is bounded since A is compact, i.e., $|f| \le \|f\|_{L^\infty(A)}$ on A. Since $\|f\|_{L^\infty(A)}$ can be thought of as a continuous (constant) function on A, this means

$$\|Tf\|_{L^\infty(B)} \le \left\| T(\|f\|_{L^\infty(A)}) \right\|_{L^\infty(B)} \le \|f\|_{L^\infty(A)} \|T(1)\|_{L^\infty(B)},$$

which establishes $\|T\| \le \|T(1)\|_{L^\infty(B)}$ and the boundedness of T. On the other hand, $\|1\|_{L^\infty(A)} = 1$, and so

$$\|T\| = \sup_{\|f\|_{L^\infty(A)}=1} \|Tf\|_{L^\infty(B)} \ge \|T(1)\|_{L^\infty(B)},$$

establishing equality.

(b)

For any $n \ge m$, $T_n - T_m \ge 0$, meaning

$$\|T_n - T_m\| = \|T_n(1) - T_m(1)\|_{L^\infty(B)}.$$

Thus, $\{T_n\}_{n=1}^\infty$ is Cauchy if and only if $\{T_n(1)\}_{n=1}^\infty$ is Cauchy in $C(B)$. However, since $C(B)$ (and, consequently, the space of all bounded linear operators from $C(A)$ to $C(B)$) are Banach spaces, this means $\{T_n\}_{n=1}^\infty$ converges if and only if $\{T_n(1)\}_{n=1}^\infty$ converges.

Problem 2

(a)

State the definition, and uniform boundedness comes from applying PUB to the evaluation maps E_{x_n}.

(b)

The weak topology on X is the smallest topology on which every $f \in X^*$ is continuous. A base for this topology at 0 is the same one described on August 2016, problem 2(a).

(c)

Clearly, the x we need is obtained by taking some kind of limit of $\{x_n\}_{n=1}^\infty$. Since this sequence is bounded and H is separable and reflexive (remember that all Hilbert spaces are reflexive), Banach-Alaoglu gives us a weakly convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$. Let x denote the weak limit of this subsequence.

We claim $Tx = f$. Indeed, since T is bounded, $Tx_{n_k} \rightharpoonup Tx$, but we are also told that $Tx_{n_k} = f_{n_k} \rightharpoonup f$. Since weak limits are unique, we must have $Tx = f$.

Problem 3

(a)

$\underline{\Longrightarrow}$ Suppose $x \in M^\perp$. The density of M means there exists a sequence $\{x_n\}_{n=1}^\infty$ in M converging to x, but this means $\|x\|^2 = \lim_{n\to\infty} \langle x_n, x\rangle = 0$, so $M^\perp = \{0\}$.

$\underline{\Longleftarrow}$ $(\overline{\mathrm{span}(M)})^\perp \subseteq M^\perp = \{0\}$ since $\perp$ reverses set inclusions. In particular, $(\overline{\mathrm{span}(M)})^\perp = \{0\}$, so

$$\overline{\mathrm{span}(M)} = ((\overline{\mathrm{span}(M)})^\perp)^\perp = \{0\}^\perp = H.$$

(remember that if Y is a closed subspace, then $(Y^\perp)^\perp = Y$. Generally, $(Y^\perp)^\perp \supseteq Y$).

(b)

Same as January 2023, problem 2(b).

August 2015

Problem 1

In a finite-dimensional space, all norms are equivalent, so Y is isomorphic and homeomorphic to $\mathbb{F}^d$. This space is complete since $\mathbb{F}$ is a complete field, so Y

is also complete as a normed linear space, and therefore closed when thought of as a subspace of X.

Suppose $Y = \text{span}\{y_1, \ldots, y_d\}$. Since $\text{span}\{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_d\}$ is finite-dimensional and therefore closed for $i = 1, \ldots, d$, Mazur's first separation lemma gives some $f_i \in X^*$ such that $f_i(y_i) = 1$ while $f_i(y_j) = 0$ for $i \neq j$. From here, set $Px = f_1(x)y_1 + \ldots + f_d(x)y_d$. Then, T is continuous and linear since all the f_i are continuous and linear, and P fixes all elements in Y.

On the other hand, if $Y = \text{span}\{y_0\}$, Hahn-Banach gives some $f \in X^*$ such that $f(y_0) = 1$. From here, define $P : X \to Y$ as $Px = f(x)y_0$. Then, P is continuous and linear since f is continuous and linear, and $P(\lambda y_0) = \lambda y_0$.

Problem 2

Same as January 2020, problem 1(b) and 1(c).

Problem 3

(a)

Let $u \in L^p$ and estimate

$$\begin{aligned}
\|Au\|_p^p &= \int_a^b \left| \int_a^t v(s)u(s)\,ds \right|^p dt \\
&\leq \int_a^b \left(\int_a^t |v(s)u(s)|\,ds \right)^p dt \\
&\leq \int_a^b \|v\|_q^p \|u\|_p^p \,dt && \text{Hölder's inequality} \\
&= (b-a) \|v\|_q^p \|u\|_p^p .
\end{aligned}$$

In particular, $\|A\| \leq (b-a)^{1/p} \|v\|_q$. This shows A maps L^p into L^p and is continuous.

(b)

First, let's assume we are working over $C(\Omega)$ only so we can use Arzelà–Ascoli. Suppose $\{u_n\}_{n=1}^\infty$ is bounded in $C(\Omega)$ under the L^∞ norm. Then,

$$\|Au_n\|_\infty = \sup_{t\in\Omega} \left| \int_a^t v(s)u(s)\,ds \right| \leq \int_a^b |v(s)u(s)|\,ds \leq (b-a)\|v\|_\infty \sup_{n\in\mathbb{N}} \|u_n\|_\infty < \infty,$$

giving us uniform boundedness. For equicontinuity, notice that if $a \leq t_1 < t_2 \leq b$, then

$$|Au_n(t_1) - Au_n(t_2)| = \left| \int_{t_1}^{t_2} v(s)u_n(s)\,ds \right| \leq (t_2 - t_1)\|v\|_\infty \sup_{n\in\mathbb{N}} \|u_n\|_\infty .$$

In particular, if $|t_1 - t_2|$ is small, then $|Au_n(t_1) - Au_n(t_2)|$ is small for any n. By Arzelà–Ascoli, $\{Au_n\}_{n=1}^{\infty}$ has a convergent subsequence, so A is compact in the $C(\Omega)$ case. Finally, by using the density of L^p and L^q functions in $C(\Omega)$, this can be extended to our original case, establishing the compactness of A.

Problem 4

(a)

Recall that positive operators on complex Hilbert spaces are self-adjoint, so for all $x, y \in X$, $\langle x, Ay\rangle = \langle Ax, y\rangle$. In particular, $\langle x, Ay\rangle = 0$ for all y if and only if $x \in [\text{Range}(A)]^{\perp}$, and $\langle Ax, y\rangle = 0$ for all y if and only if $x \in \text{Null}(A)$.

(b)

All spectral values of positive operators are nonnegative by ST(sa), so $I + tA = t(A-(-t^{-1}A)) = tA_{-t^{-1}}$ is bijective since $-t^{-1} < 0$ and therefore not a spectral value.

(c)

First, suppose $x \in \text{Null}(A)$. Since $(I + tA)x = x$ for all $t > 0$, we also have $(I + tA)^{-1}x = x$. Thus, $\lim_{t\to\infty}(I + tA)^{-1}x \to x$.

By part (a), we have

$$[\text{Null}(A)]^{\perp} = ([\text{Range}(A)]^{\perp})^{\perp} = \overline{\text{Range}(A)}$$

(the second equality is exercise 3.15(a) of Arbogast and is not hard to prove). We will first show that $\lim_{t\to\infty}(I + tA)^{-1} = 0$ on Range(A). Now, since A commutes with I and tA, it also commutes with $I + tA$ and therefore commutes with $(I + tA)^{-1}$. Thus,

$$(I + tA)^{-1}A = \frac{1}{t}(At)(I + tA)^{-1}$$

If s is a nonzero real number, then

$$\frac{ts}{1+ts} = \frac{1+ts-1}{1+ts} = 1 - \frac{1}{1+ts}.$$

With this as inspiration, it is not hard to check that $(tA)(I+tA) = I-(I+tA)^{-1}$. Hence,

$$\frac{1}{t}(At)(I + tA)^{-1} = \frac{1}{t}[I - (I + tA)^{-1}].$$

From here, we have for any $x \in X$

$$\begin{aligned}\left\|(I + tA)^{-1}Ax\right\| &= \frac{1}{t}\left\|(I - (I + tA)^{-1})x\right\| \\ &\le \frac{1}{t}\left[\|x\| + \left\|(I + tA)^{-1}x\right\|\right].\end{aligned}$$

From here, we can estimate the norm on the right. Letting $y = (I + tA)^{-1}x$, we have

$$\langle x, y \rangle = \langle y + tAy, y \rangle = \|y\|^2 + t \langle Ay, y \rangle \geq \|y\|^2$$

since A is positive, meaning

$$\|y\|^2 \leq \langle x, y \rangle \leq \|x\| \|y\|$$

By Cauchy-Schwartz. Thus, $\|y\| \leq \|x\|$, which implies

$$\left\|(I + tA)^{-1}Ax\right\| \leq \frac{2\|x\|}{t}.$$

In particular, this goes to 0 as $t \to \infty$, meaning $\lim_{t\to\infty}(I + tA)^{-1} = 0$ on $\mathrm{Range}(A)$. By continuity, this is also true over $[\mathrm{Null}(A)]^{\perp}$. To conclude, just use the fact that $X = \mathrm{Null}(A) \oplus [\mathrm{Null}(A)]^{\perp}$ to find that the limit is $P_{\mathrm{Null}(A)}$.

Possible Prelim Problems

The following are results from the book whose proofs have not been asked on a previous prelim, but I believe have a decent chance of going on a prelim (or, at least parts of the proof).

Proposition 2.11

Statement: Let X be a finite-dimensional vector space. Then, all norms on X are equivalent, and a subset of X is compact if and only if it is closed and bounded.

Proof. Taking two arbitrary norms on X and trying to show they are equivalent is a trap. Instead, look to define a "nice" norm on X, then show every norm is equivalent to this nice one. Here's how to do it:

Let $e_1, \ldots, e_n$ denote a basis of X. From here, define $T : X \to \mathbb{F}^n$ by

$$Tx = T(x_1e_1 + \ldots x_ne_n) = (x_1, \ldots, x_n).$$

From here, define this "nice" norm on X:

$$\|x\|_1 = \|Tx\|_{\ell^1}.$$

It is easy to check the following important facts:

- T is linear and a bijection
- $\|\cdot\|_1$ is a norm on X
- T is an isometry with respect to the ℓ^1 norm on $\mathbb{F}^n$ and the $\|\cdot\|_1$ norm on X

- T^{-1} is also an isometry
- T is a homeomorphism

From here, let S_1^1 denote the closed unit ball in X under the norm $\|\cdot\|_1$. T sends this set to the closed unit ball in $\mathbb{F}^n$, which is compact by Heine-Borel. Thus, S_1^1 is compact.

Next, let $\|\cdot\|$ be any arbitrary norm over X. The first inequality is easy:

$$\|x\| = \left\| \sum_{i=1}^{n} x_i e_i \right\| \leq \sum_{i=1}^{n} |x_i| \cdot \|e_i\| \leq C \|x\|_1 ,$$

where $C = \max_{1 \leq i \leq n} \|e_i\|$ (note that C is positive, for otherwise $\|\cdot\|$ would be identically zero). This calculation also reveals that every open ball under $\|\cdot\|_1$ is contained in an open ball under $\|\cdot\|$ (in particular, a ball of radius r under $\|\cdot\|_1$ is contained in a ball of radius $C^{-1}r$ under $\|\cdot\|$). This tells us that the topology of $(X, \|\cdot\|)$ is contained in the topology of $(X, \|\cdot\|_1)$. In particular, S_1^1 is compact under the topology of $(X, \|\cdot\|)$. Now, defining $a = \inf_{x \in S_1^1} \|x\|$, we must have that $a > 0$ since the continuity of $\|\cdot\|$ and the compactness of S_1^1 means a minimum is attained, and this minimum must have nonzero norm since it lies on the unit sphere. This gives us our final estimate:

$$a \|x\|_1 \leq \left\| \frac{x}{\|x\|_1} \right\| \|x\|_1 = \|x\|$$

for all $x \neq 0$ since $\dfrac{x}{\|x\|_1} \in S_1^1$. Thus, $\|x\|_1 \leq a^{-1} \|x\|$ for all $x \in X$. <u>QED</u>

Theorem 2.28 (Extending Hahn-Banach from $\mathbb{R}$ to $\mathbb{C}$)

<u>Statement</u>: Hahn-Banach works if $\mathbb{C}$ is the field, assuming we know it works over $\mathbb{R}$.

Proof. Start with a subspace $Y \subseteq X$ and a linear functional f over Y (with $\mathbb{C}$ as the field!) such that $|f| \leq p$ on Y for some seminorm p. Then, break f into its real and imaginary parts, which we will denote g and h respectively. Notice that

$$g(ix) + ih(ix) = f(ix) = if(x) = ig(x) - h(x).$$

Compare the real parts to conclude that $g(ix) = -h(x)$. Thus, $f(x) = g(x) - ig(ix)$ (get used to this "comparing real parts" trick – we will use it multiple times in this proof). Now, g can be thought of as a linear functional on Y but with $\mathbb{R}$ as the field, since for $\lambda \in \mathbb{R}$,

$$f(\lambda x) = g(\lambda x) + ih(\lambda x),$$
$$\lambda f(x) = \lambda g(x) + i\lambda h(x).$$

Once again, we recognize that the top and bottom lines are equal, and comparing real parts tells us $g(\lambda x) = \lambda g(x)$. Also, $|g| \leq |f| \leq p$ on Y. Thus, Hahn-Banach over $\mathbb{R}$ lets us extend g to some linear functional G over X (remember that G is only real-linear and real-valued) such that $|G| \leq p$ on X. Now, define the *complex*-valued functional $F(x) = G(x) - iG(ix)$ over X. Clearly, F extends f. For linearity, we really only need to check that i factors out:

$$F(ix) = G(ix) - iG(-x) = iG(x) + G(ix) = i[G(x) - iG(ix)] = iF(x).$$

(remark: we do not know if $G(ix) = iG(x)$. Regardless, things still work out since we only need that F is complex-linear). Finally, we need to check that $|F| \leq p$ on X.

<u>Trick</u>: Use polar decomposition. Fix $x \in X$ and write $F(x) = re^{i\theta}$. Then,

$$|F(x)| = r = e^{-i\theta}F(x) = F(e^{-i\theta}x) = G(e^{-i\theta}x) - iG(e^{-i\theta}x).$$

But now, the right hand side must be a nonnegative real number, so $iG(e^{-i\theta}x) = 0$ and $G(e^{-i\theta}x) \geq 0$. This tells us

$$|F(x)| = \left|G(e^{-i\theta}x)\right| \leq p(e^{-i\theta}x) = \left|e^{-i\theta}\right| p(x) = p(x).$$

<u>QED</u>

Important thing to note: this general formulation of Hahn-Banach does not require our linear functional to be bounded in operator norm. However, most traditional applications use it on bounded linear functionals.

Theorem 2.45 (Closed Graph Theorem)

<u>Statement</u>: If X and Y are Banach spaces and $T : X \to Y$ linear, then T is continuous if and only if it is closed.

Proof. $\underline{\implies}$ This can be generalized into a broader statement on Hausdorff spaces, but let's not bother. Let (x_n, Tx_n) be a sequence in Graph(T) converging to some $(x, y) \in X \times Y$. Then, $x_n \to x$ and $Tx_n \to y$. But, by the continuity of T, $Tx_n \to Tx$, and so $Tx = y$, making $(x, y) = (x, Tx) \in$ Graph(T).

$\underline{\impliedby}$ Suppose T is closed. Then, Graph(T) is a closed subspace of $X \times Y$, and therefore a Banach space in its own right. In particular, the norm on Graph(T) is simply $\|(x, Tx)\| := \|x\|_X + \|Tx\|_Y$.

<u>Trick</u>: Look at the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : \text{Graph}(T) \to X \times Y$. We will leverage the fact that these are continuous. In particular, their restrictions to Graph(T) are continuous, and clearly π_1 is a bijection when restricted to Graph(T). Thus, its inverse $\pi_1^{-1} : X \to \text{Graph}(T)$ given by $\pi^{-1}(x) = (x, Tx)$ is continuous by OMT. But now, $T = \pi_2 \circ \pi_1^{-1}$, which is continuous being a composition of continuous functions. <u>QED</u>

Theorem 2.47 (Uniform Boundedness Principle)

Statement (as in the book): X Banach, Y NLS, $\{T_\alpha\}_{\alpha\in\mathcal{I}}$ a collection of bounded linear operators from X to Y. Either this collection is uniformly bounded, or unbounded at a point.

Proof. Trick: Let $\varphi_\alpha := \|T_\alpha\|$ for all α (notice φ_α is continuous from X to $[0,\infty)$). Then, define the open sets

$$V_n = \bigcup_{\alpha\in\mathcal{I}} \varphi_\alpha^{-1}((n,\infty))$$

for all $n\in\mathbb{N}$. There are now two cases to consider. First, suppose there exists N such that V_N is not dense in X. Then, there exists an open ball $B_r(x)$ that does not intersect V_N. In particular, for all $z\in B_r(x)$ and $\alpha\in\mathcal{I}$, $\|T_\alpha z\|\le N$. It follows that for all α,

$$\|T_\alpha\| = \sup_{\|y\|\le 1}\|T_\alpha y\| = \frac{2}{r}\sup_{\|y\|\le 1}\left\|T_\alpha\left(\frac{ry}{2}+x\right)\right\| + \|T_\alpha x\| \le \frac{4N}{r}$$

since $\dfrac{ry}{2}+x\in B_r(x)$ whenever $\|y\|\le 1$. Hence, this case implies we have uniform boundedness.

For the other case, suppose all the V_n are dense in X. By Baire Category Theorem, $\bigcap_{n=1}^\infty V_n$ is dense in X, and in particular, nonempty. Let $x\in\bigcap_{n=1}^\infty V_n$. Then, for all $n\in\mathbb{N}$, there exists $\alpha_n\in\mathcal{I}$ such that $\|T_{\alpha_n}x\|>n$, meaning the collection $\{T_\alpha\}_{\alpha\in\mathcal{I}}$ is unbounded at x. QED

Lemma 2.49 (Separation from a Closed Subspace)

Statement: X and NLS, Y a closed subspace, and Z a subspace containing Y. If $Z\neq Y$ and $\theta\in(0,1)$, there exists $z\in Z$ such that $\|z\|=1$ and $\operatorname{dist}(z,Y)\ge\theta$.

Proof. Fix $z_0\in Z\setminus Y$ and put $d=\operatorname{dist}(z,Y)$. Since Y is closed, $d>0$. Thus, there exists $y_0\in Y$ such that $\|z_0-y_0\|<\dfrac{d}{\theta}$.

Trick: Let $z := \dfrac{z_0-y_0}{\|z_0-y_0\|}$ (clearly, $z\in Z$ and $\|z\|=1$). Then, for all $y\in Y$,

$$\|z-y\| = \frac{1}{\|z_0-y_0\|}\cdot\|z_0-y_0-\|z_0-y_0\|\,y\| > \frac{\theta}{d}\cdot d = \theta$$

since $y_0-\|z_0-y_0\|\,y$ is an element of Y, so its distance to z_0 must be greater than d. QED

Corollary 2.50 (No Infinite-Dimensional Heine-Borel)

Statement: The closed unit ball in an infinite-dimensional NLS is not compact.

Proof. Let X be infinite-dimensional and B the closed unit ball in X. Since we are working over an NLS, it is enough to produce a sequence in B with no convergent subsequence (i.e., we need all the points in our sequence to be far away from each other). This is where we deploy lemma 2.49. Since X is infinite-dimensional, we have a linearly independent sequence $\{y_n\}_{n=1}^{\infty}$ in B. Define $Y_n := \text{span}(y_1, \ldots, y_n)$ for all n. Then, use the following induction argument:

Since Y_1 is a closed subspace strictly contained in B, the previous lemma gives us some $x_1 \in B \setminus Y_1$ such that $\text{dist}(z, Y_1) \geq \dfrac{1}{2}$. Now, let $Y_2 = \text{span}\{Y_1, x_1\}$. Once again, this is a closed subspace strictly contained in B, so we can similarly find $x_2 \in B \setminus Y_2$ such that $\text{dist}(z, Y_2) \geq \dfrac{1}{2}$. Proceed inductively, giving us a sequence of points in the ball that are all $\dfrac{1}{2}$ away from each other. In particular, there is no convergent subsequence, so no compactness. QED

NB: *The proofs of lemma 2.49 and corollary 2.50 actually showed up on the January 2026 prelim!*

Theorem 2.62 (Banach-Saks)

Statement: X an NLS and $\{x_n\}_{n=1}^{\infty}$ a sequence converging weakly to x. For every $n \geq 1$, there exist constants $\alpha_j^{(n)} \geq 0$ with $\sum_{j=1}^{n} \alpha_j^{(n)} = 1$ such that $y_n := \sum_{j=1}^{n} \alpha_j^{(n)} \to x$ strongly as $n \to \infty$.

Proof. The statement seems daunting to prove, but it's really equivalent to saying that x is contained in the closure of the convex hull of $\{x_n\}_{n=1}^{\infty}$. With this reframing, we can use the previously built results that stemmed from Hahn-Banach. Let H denote the convex hull of $\{x_n\}_{n=1}^{\infty}$ and suppose for the sake of contradiction that x is not in $\overline{H}$. Then, $\overline{H}$ is closed and convex while $\{x\}$ is compact, convex, and disjoint from $\overline{H}$, so by the separating hyperplane theorem (specifically, part (c) of this theorem), there exists some $f \in X^*$ and $\gamma \in \mathbb{R}$ such that $\Re f(x) < \gamma < f(y)$ for all $y \in \overline{H}$. In particular, $\Re f(x) < \gamma < \Re f(x_n)$ for all n, but this is a contradiction since we must have $f(x_n) \to f(x)$. QED

Lemma 3.2 (Cauchy-Schwarz Inequality)

Statement: H a Hilbert space, $x, y \in H$. Then, $|\langle x, y\rangle| \leq \|x\| \, \|y\|$.

Proof. This is trivial if y is zero, so assume $y \neq 0$. Start by letting $\lambda \in \mathbb{F}$ and investigating the norm of $x - \lambda y$:

$$\begin{aligned} 0 &\leq \|x - \lambda y\|^2 \\ &= \|x\|^2 - \overline{\lambda} \langle x, y \rangle - \lambda \langle y, x \rangle + |\lambda|^2 \|y\|^2 . \end{aligned}$$

<u>Trick</u>: Take $\lambda = \dfrac{\langle x, y \rangle}{\|y\|^2}$ (the idea is to get the two middle terms to agree while canceling out a $\|y\|^2$ on the last term). This gives us

$$0 \leq \|x\|^2 - \frac{2 |\langle x, y \rangle|^2}{\|y\|^2} + \frac{|\langle x, y \rangle|^2}{\|y\|^2} = \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}.$$

Rearrange to get the desired inequality. <u>QED</u>

Corollary 3.9 (Orthogonal Projection $\implies$ Orthogonal)

<u>Statement</u>: H Hilbert and M a complete linear subspace. If $x \in H$ and $y \in M$ is the best approximation of x in M, then $x - y \perp M$.

Proof. Let $0 \neq m \in M$. We have

$$\begin{aligned} \|x - y\|^2 &\leq \|(x - y) - \lambda m)\|^2 \\ &= \|x - y\|^2 - \overline{\lambda} \langle x - y, m \rangle - \lambda \langle m, x - y \rangle + |\lambda|^2 \|m\|^2 \\ &\implies 0 \leq -\overline{\lambda} \langle x - y, m \rangle - \lambda \langle m, x - y \rangle + |\lambda|^2 \|m\|^2 . \end{aligned}$$

<u>Trick</u>: Take $\lambda = \dfrac{\langle x - y, m \rangle}{\|m\|^2}$ (same idea as the last proof!). Then,

$$0 \leq -\frac{|\langle x - y \rangle|^2}{\|m\|^2},$$

which is only possible if $\langle x - y, m \rangle = 0$. <u>QED</u>

Lemma 3.28 (Orthonormal Bases are all you Need)

<u>Statement</u>: If $\{e_\alpha\}$ is an ON basis of H and $x_n, x \in H$, then $x_n \rightharpoonup x$ if and only if $\{x_n\}_{n=1}^{\infty}$ is bounded and $\langle x_n, e_\alpha \rangle \to \langle x, e_\alpha \rangle$ for all α.

Proof. <u>$\implies$</u> Obvious

<u>$\impliedby$</u> The second hypothesis immediately implies that $\langle x_n, y \rangle \to \langle x, y \rangle$ for all $y \in \operatorname{Span}\{e_\alpha\}_{\alpha \in \mathcal{I}}$. Let $z \in H$ and $\varepsilon > 0$. Since $\{e_\alpha\}$ is an ON basis, its span is dense in H, so there exists $y \in \operatorname{Span}\{u_\alpha\}_{\alpha \in \mathcal{I}}$ such that $\|z - y\| < \varepsilon$. Now,

$$|\langle x_n - x, z \rangle| = |\langle x_n - x, z - y \rangle| + |\langle x_n - x, y \rangle| \leq C\varepsilon + |\langle x_n - x, y \rangle| \to C\varepsilon,$$

where $C := \sup_{n \in \mathbb{N}} \|x_n\| + \|x\|$. <u>QED</u>

Lemma 4.2 and Corollary 4.3 (Resolvents in Banach Spaces)

Statement: X a Banach space, $T \in B(X, X)$. Then, $\lambda \in \rho(T)$ if and only if T_λ is a bijection.

Proof. $\Longrightarrow$ We only need to check for surjectivity. By assumption, $T_\lambda^{-1} : R(T_\lambda) \to X$ is bounded. Let $y \in X$. Since T_λ has a dense range, there exists $\{x_n\}_{n=1}^\infty$ in X such that $T_\lambda x_n \to y$. Then, $\{x_n\}_{n=1}^\infty$ is Cauchy in X, since

$$\|x_n - x_m\| = \left\|T_\lambda^{-1}(T_\lambda(x_n - x_m))\right\| \leq \left\|T_\lambda^{-1}\right\| \|T_\lambda x_n - T_\lambda x_m\| \to 0.$$

Thus, $\{x_n\}_{n=1}^\infty$ converges to some $x \in X$. But now, the continuity of T_λ tells us that $T_\lambda x_n \to T_\lambda x$, and so $y = T_\lambda x$, establishing surjectivity.

$\Longleftarrow$ Of course, T_λ is one-to-one with a dense range (since its range is all of X). Lastly, OMT tells us T_λ^{-1} is bounded, so λ is in the resolvent. QED

Lemma 4.4 (Geometric Series for "Small" Operators)

Statement: X Banach, $V \in B(X, X)$ with $\|V\| < 1$. Then, $I - V$ is invertible with $(I - V)^{-1} = \sum_{n=0}^\infty V^n$.

Proof. Define the partial sums $S_N := \sum_{n=0}^N V^n$. This sequence is Cauchy in $B(X, X)$, since if $N > M$, then

$$\|S_N - S_M\| = \left\|\sum_{n=M+1}^N V^n\right\| \leq \sum_{n=M+1}^N \|V\|^n.$$

But $\|V\| < 1$, meaning $\sum_{n=0}^\infty \|V\|^n$ is a convergent geometric series. In particular, the right hand side tends to 0 as $N, M \to \infty$, so the partial sums form a Cauchy sequence. But $B(X, X)$ is complete, so S_N has a limit that we will denote by S. First, notice that

$$S_N(I - V) = (I - V)S_N = I - V^{N+1},$$

and since

$$\left\|V^{N+1}\right\| \leq \|V\|^{N+1} \to 0,$$

we will have $S_N(I - V) \to I$ in operator norm. However, it is also true that $S_N(I - V) \to S(I - V)$, and so $S(I - V) = I$. Likewise, $(I - V)S = I$, so $I - V$ is invertible with $(I - V)^{-1} = S$, as desired.

QED

Proposition 4.16 (Compact Operators have Finite-Dimensional Eigenspaces)

Statement: X Banach, $T \in C(X, X)$. For all $\lambda \neq 0$, $N(T_\lambda)$ is finite-dimensional.

Proof. Of course, if λ is not an eigenvalue, then $N(T_\lambda) = \{0\}$, so we can assume $\lambda \in \sigma_p(T)$. Our strategy here is to show that the closed unit ball in $N(T_\lambda)$ is compact, since this is only possible if we are in a finite-dimensional space. So, let $B = \overline{B_1(0)} \cap N(T_\lambda)$, and let $\{x_n\}_{n=1}^\infty$ be a sequence in B. In particular, $x_n = T\left(\dfrac{x_n}{\lambda}\right)$ for all n. But $\{x_n/\lambda\}_{n=1}^\infty$ is a bounded sequence as well, so $\{x_n\}_{n=1}^\infty$ has a convergent subsequence since T is compact; the limit of this sequence is in B since B is closed. QED

Probably Not Prelim Material

The following are Applied I topics that have not shown up on prelims in any form for the past 10 years. In my opinion, these are unlikely to show up on future prelims either because they are too time-consuming or unrelated to the core content, but ignore these at your own risk.

- Proof of OMT
- Proof of Mazur's second separation lemma
- Proof of separating hyperplane theorem
- Proof of BAT (since it requires Tychonoff's theorem)
- Any proof that requires Zorn's Lemma (so, HBT for vector spaces over $\mathbb{R}$ and the proof of the existence of an ON basis)
- Proof of L^p Duality Theorem (since it requires Radon-Nikodym)
- Proof of ST(c)
- Proof of Hilbert-Schmidt Theorem
- Sturm-Liouville theory
- Distributional solutions to differential equations (ODE and PDE)
- Distributional convolutions
- Leibniz rule
- Finding distributional derivatives
- Approximations to the identity